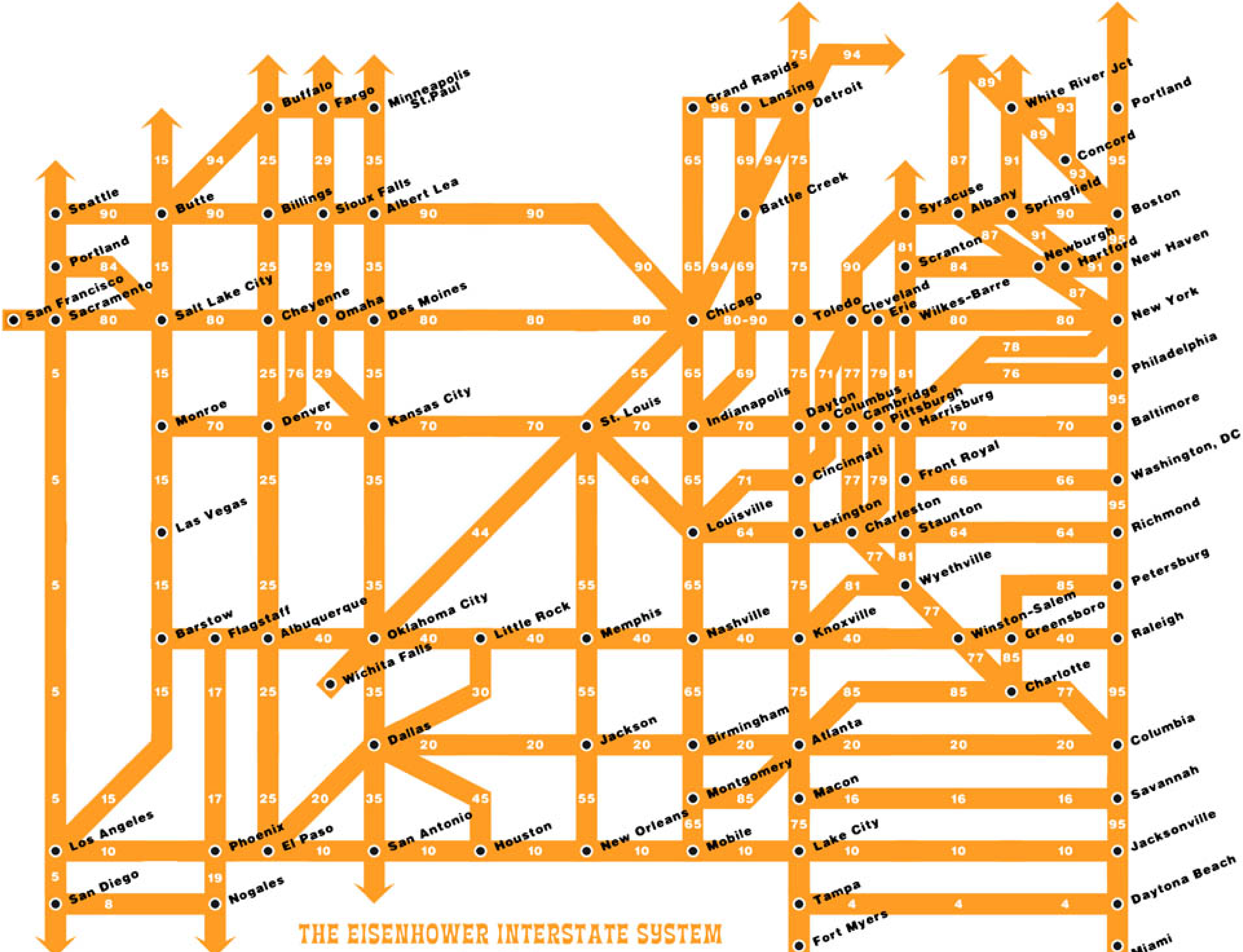
Graph Theory

Part One

Outline for Today

- ***Graphs and Digraphs***
  - Two fundamental mathematical structures.
- ***Graphs Meet FOL***
  - Building visual intuitions.
- ***Independent Sets and Vertex Covers***
  - Two structures in graphs.

Graphs and Digraphs

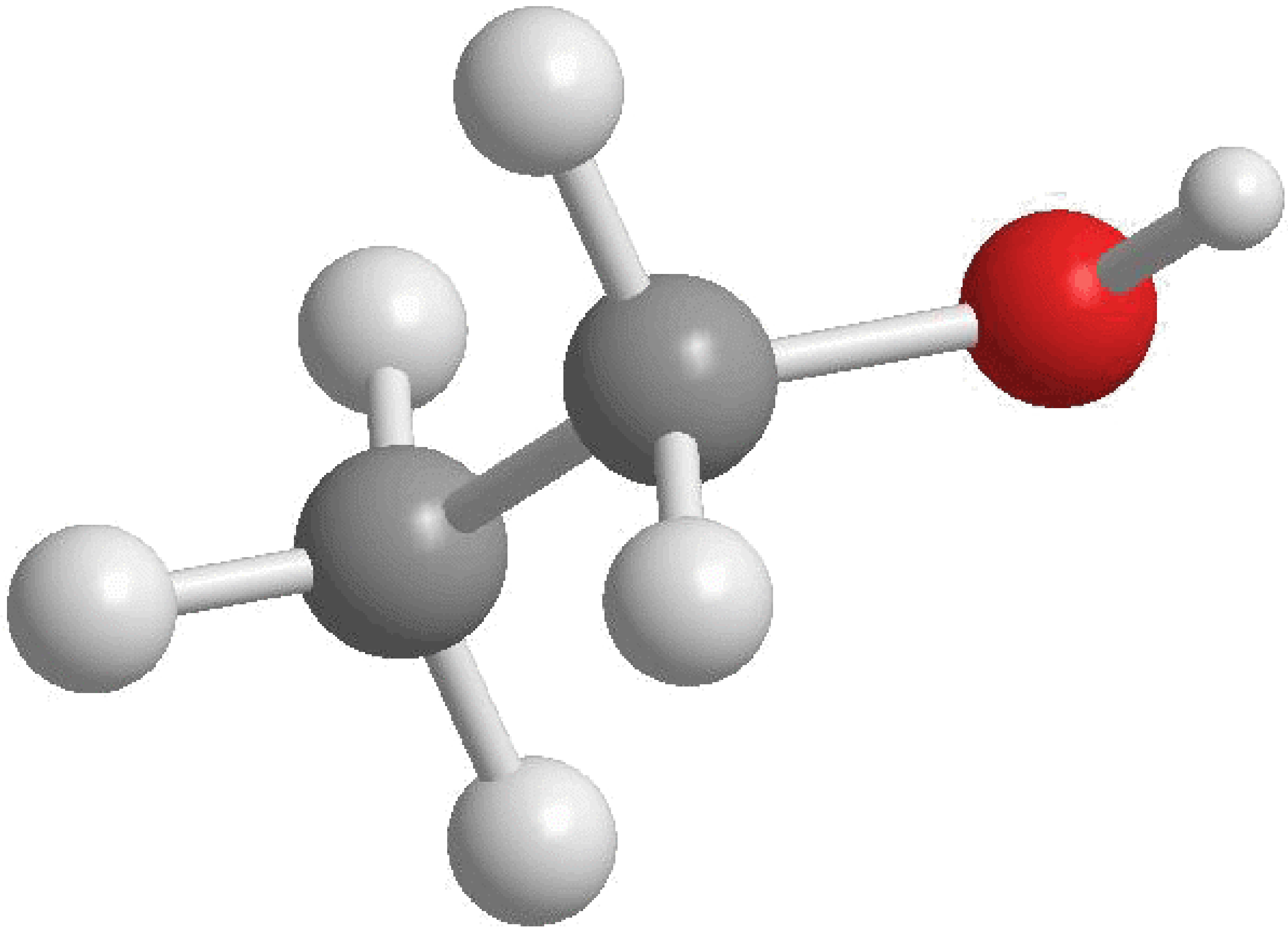


THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007

Chemical Bonds



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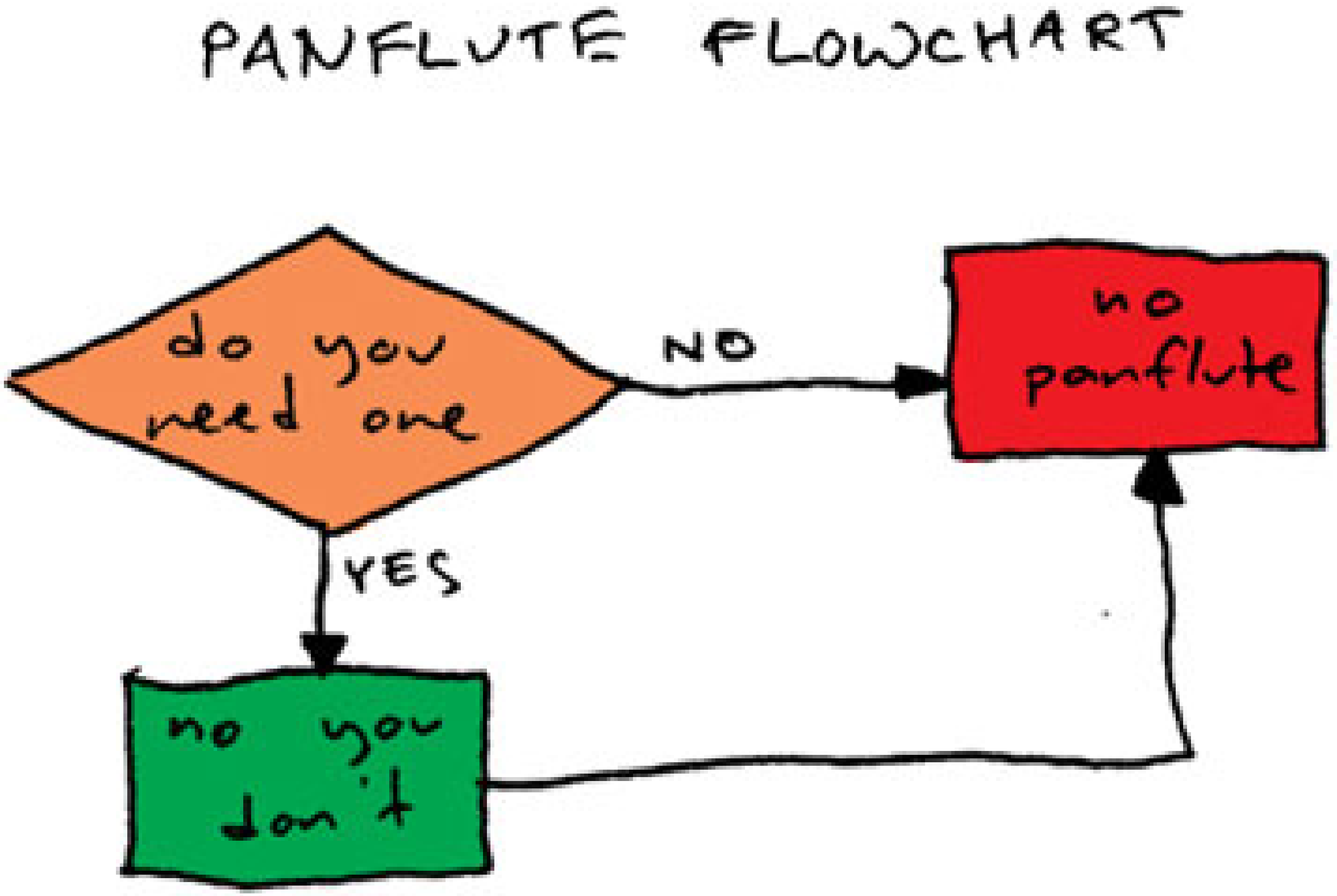
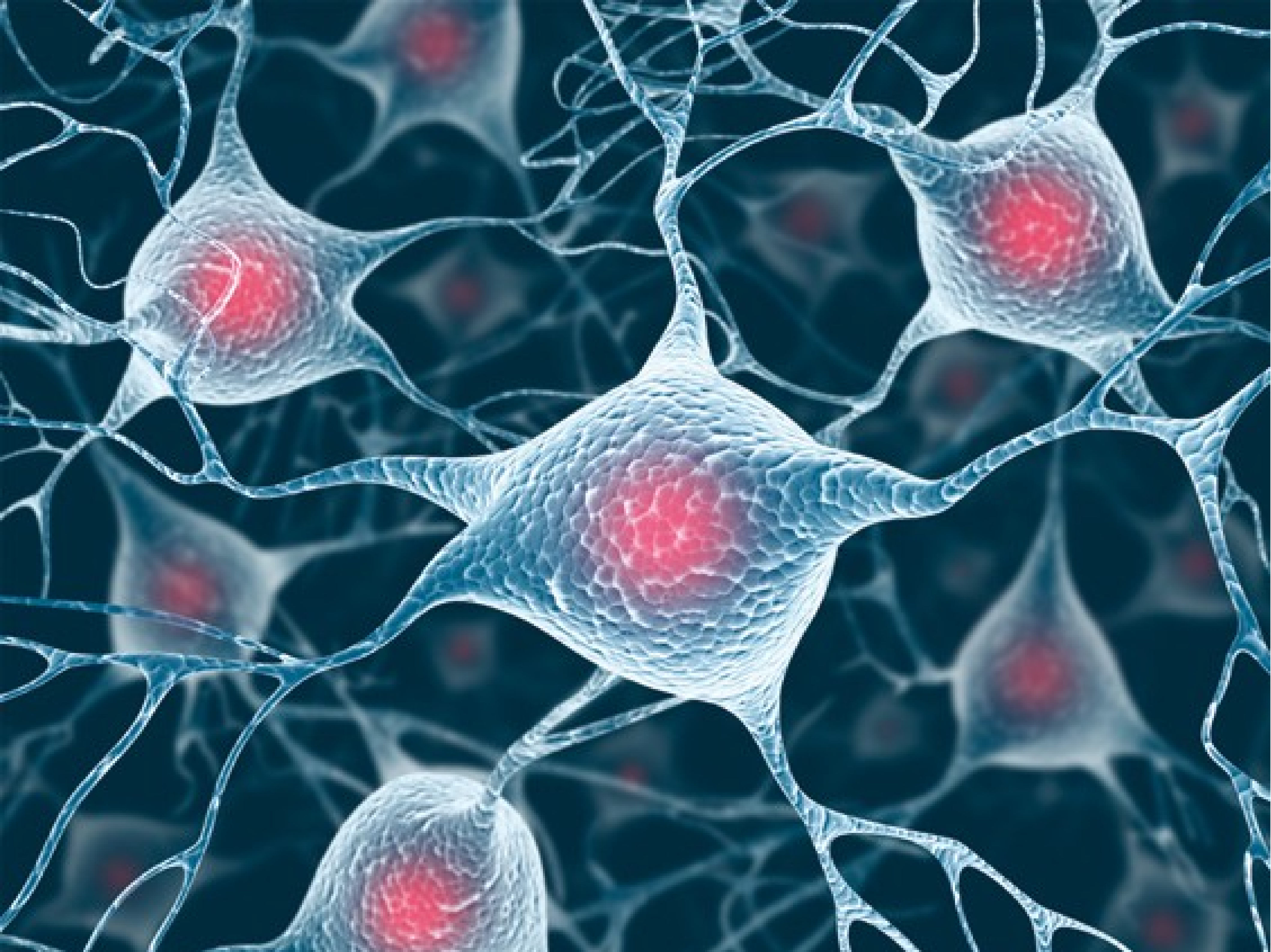

PANFLUTE FLOWCHART
do you need one
NO
no panflute
YES
no you don't



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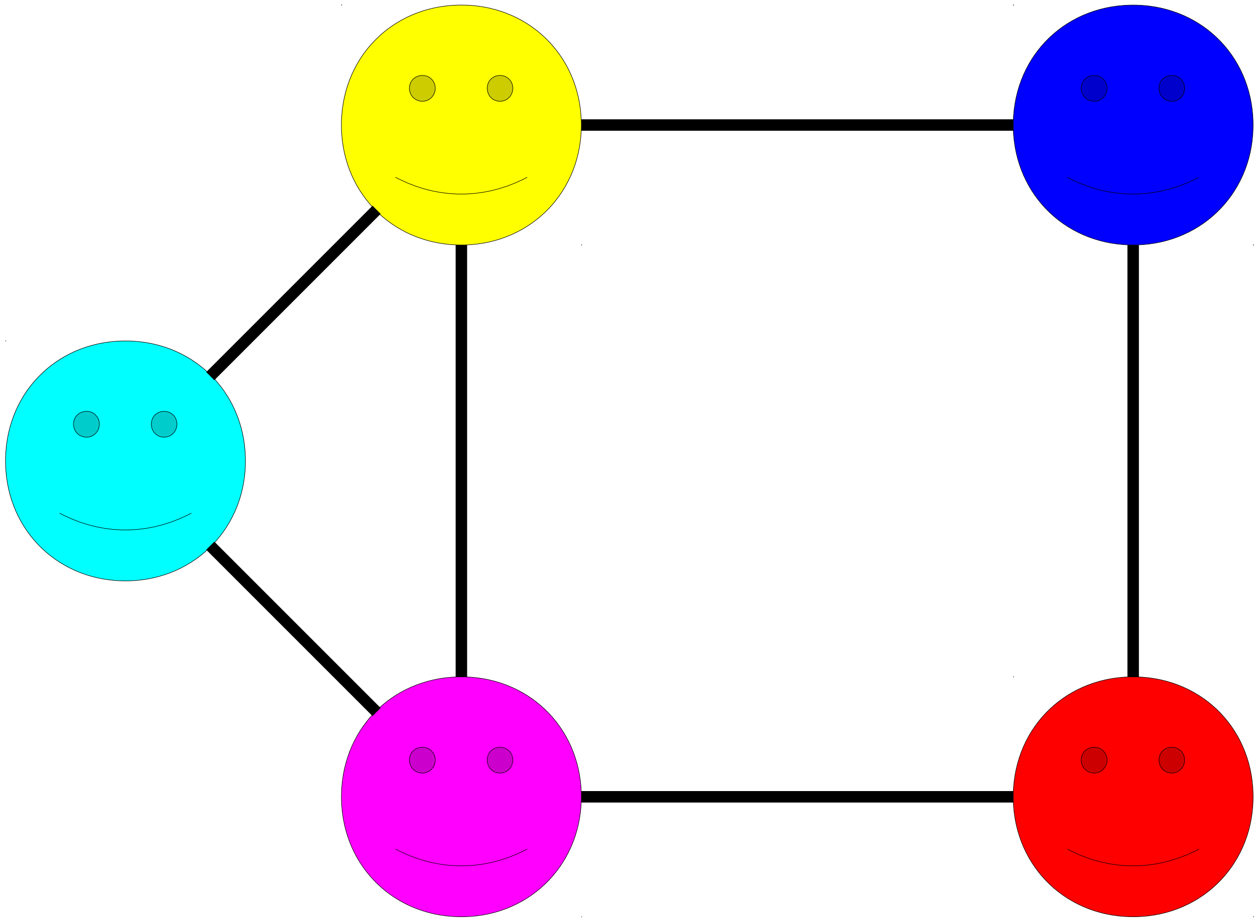
What's in Common

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- ***Goal:*** find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.



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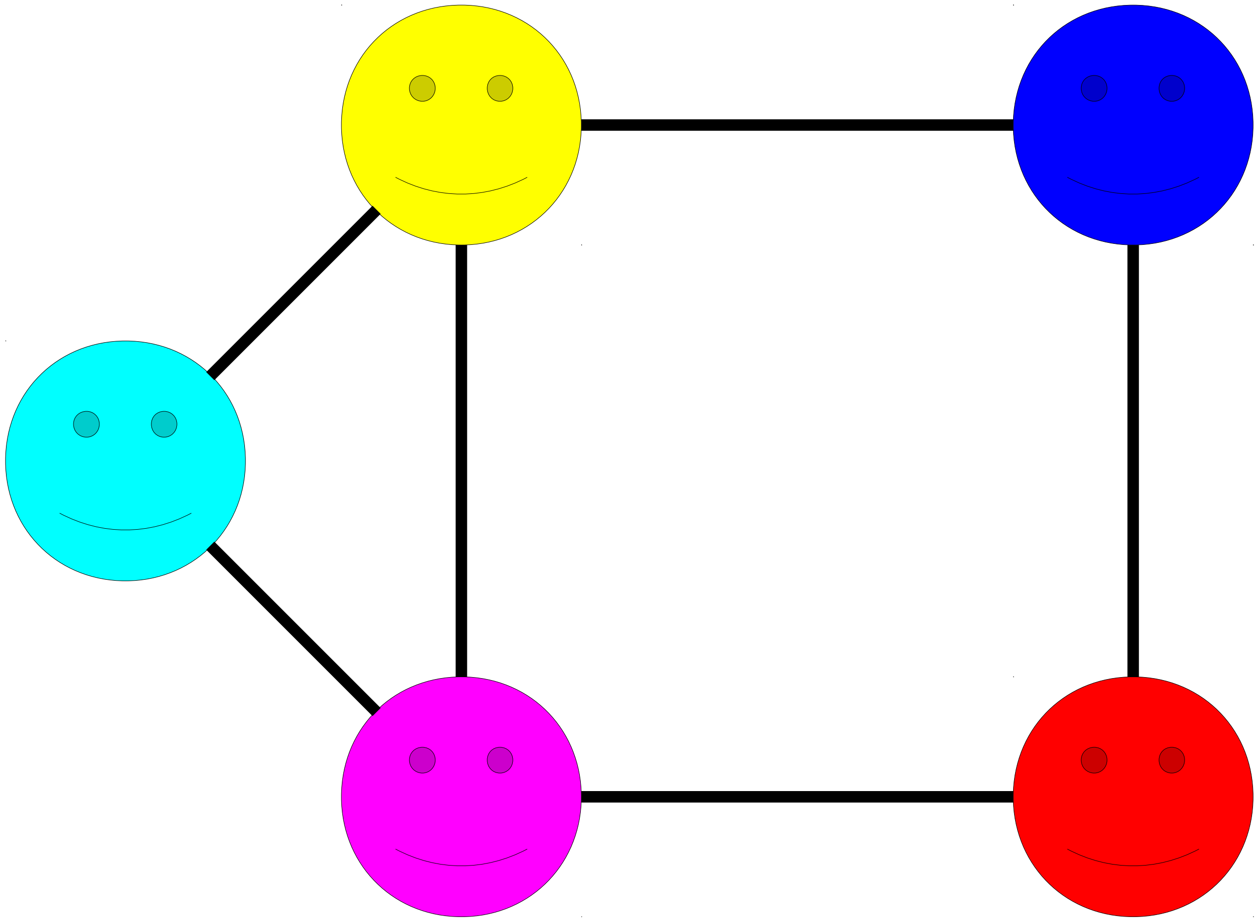

A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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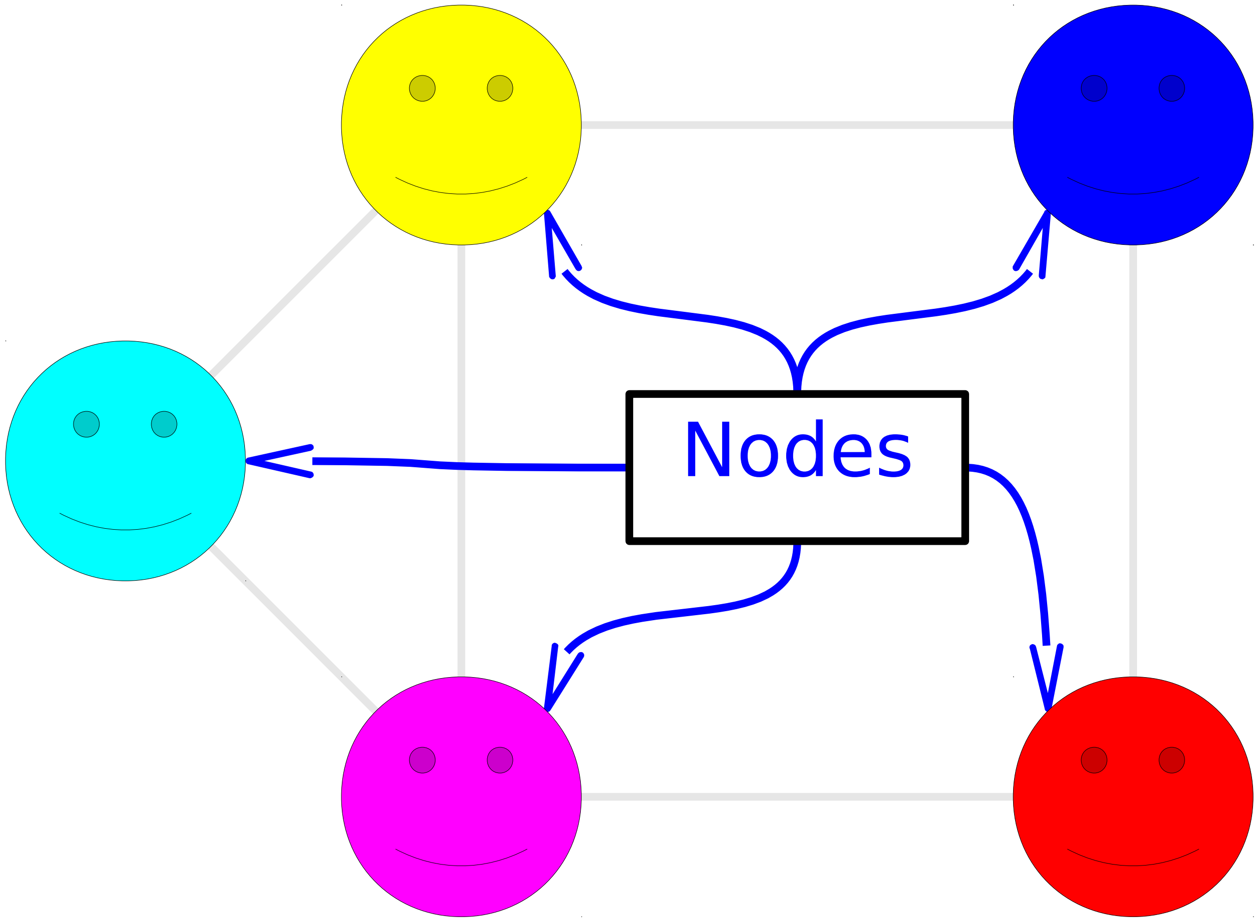

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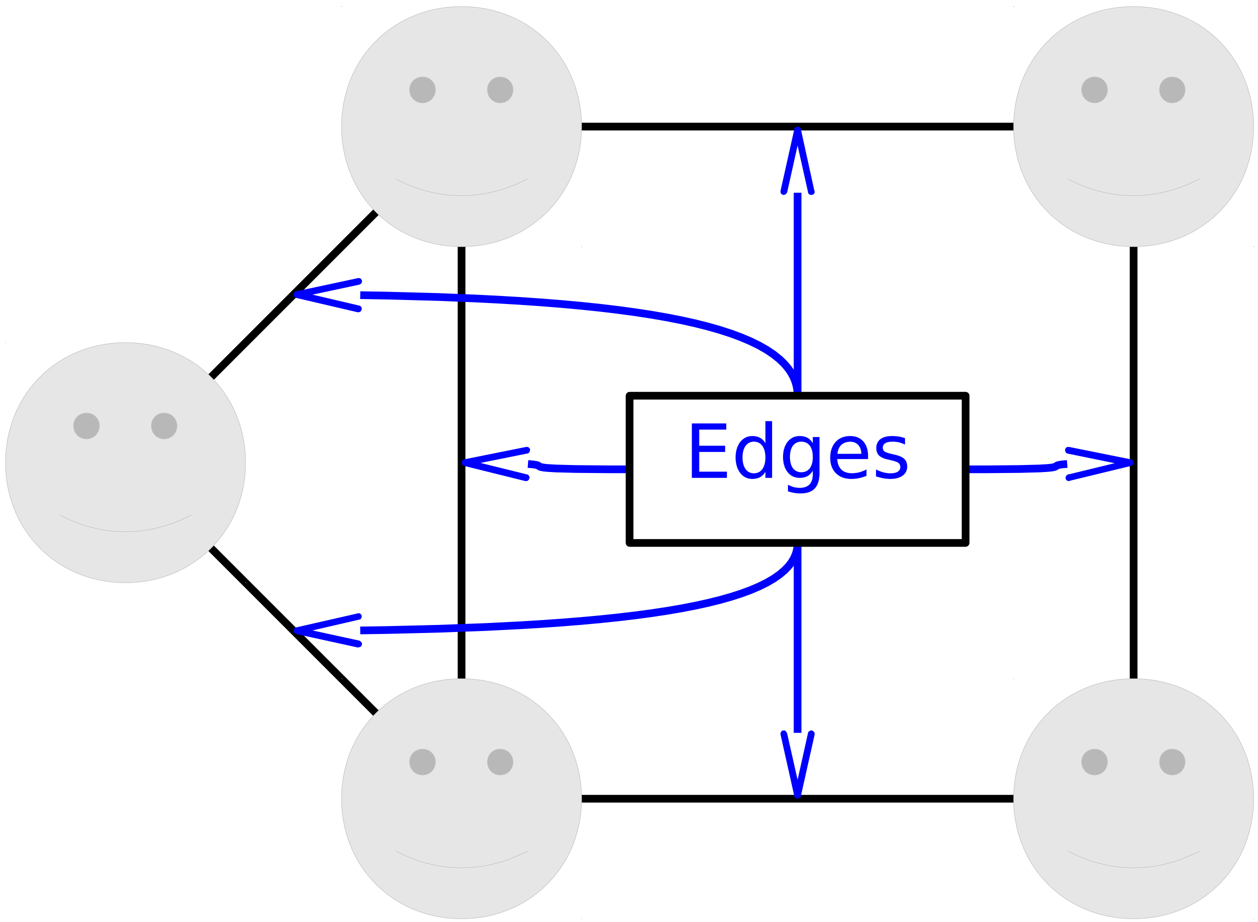

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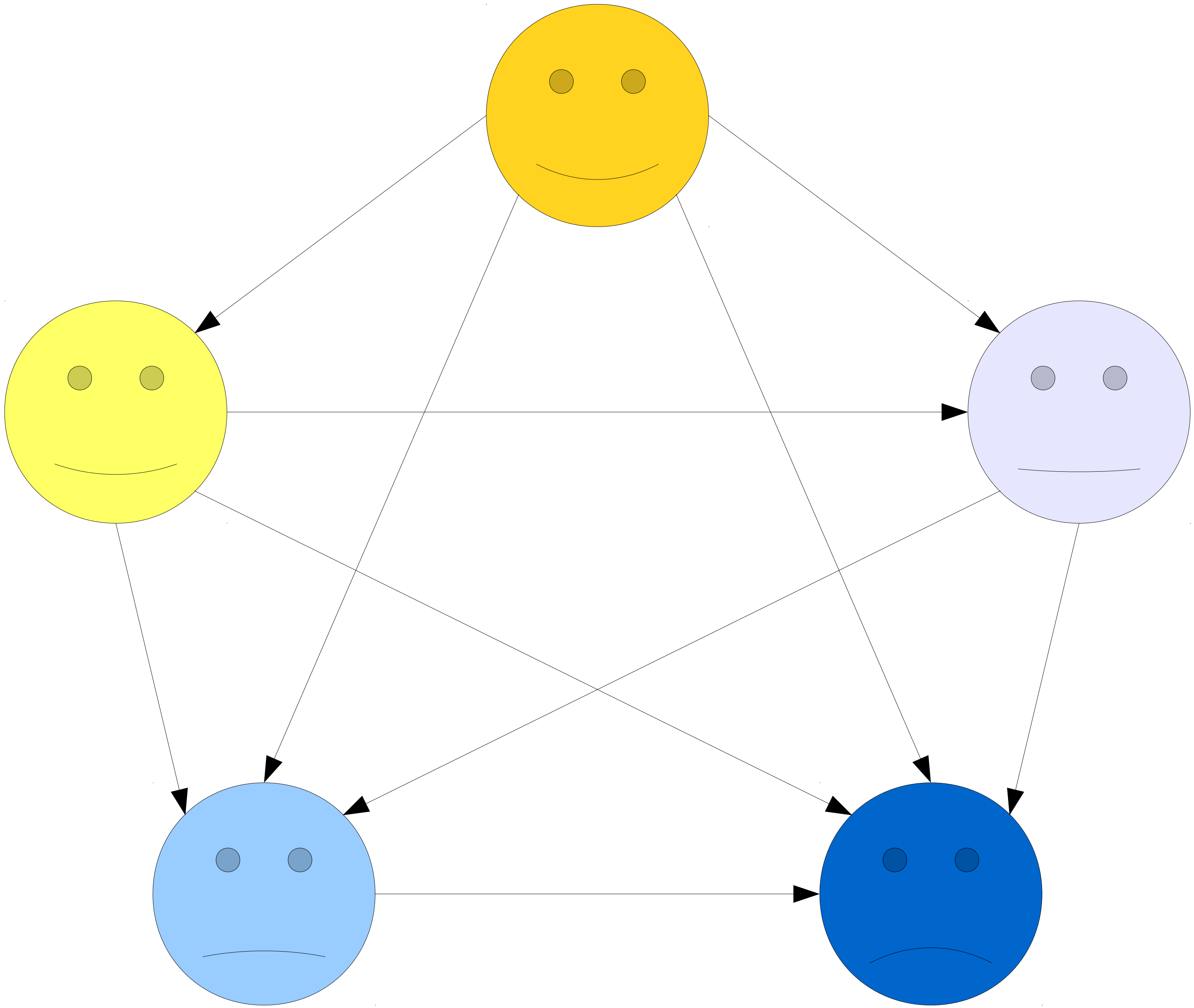
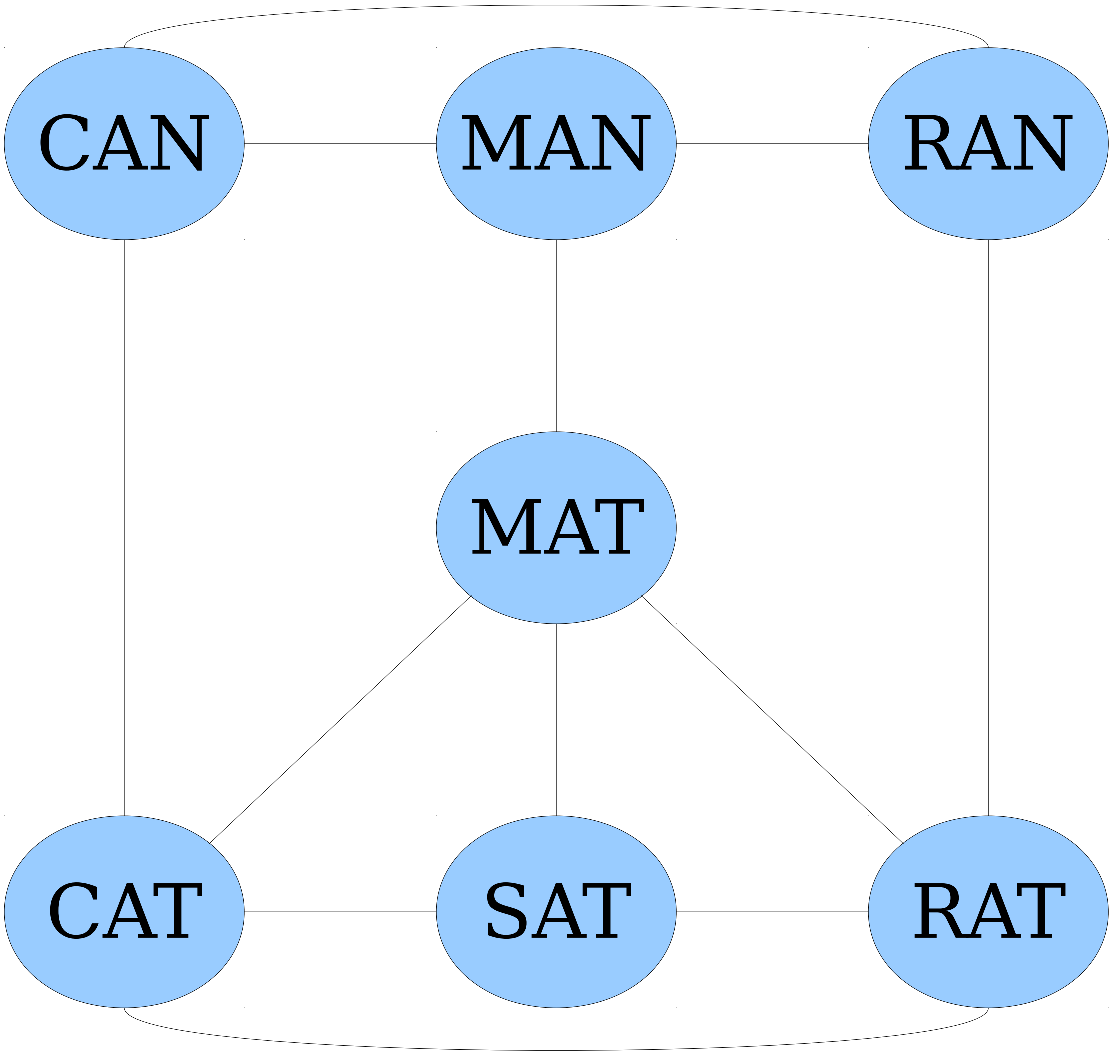
Some graphs are ***directed***.



Some graphs are ***undirected***.



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.

- Unless specified otherwise:

  ☞ ***"Graph" means "undirected graph"*** ☜

Formalizing Graphs

- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
  - An ***unordered pair*** is a set with cardinality two.
- We won't use them in this class, but a ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *ordered* pairs of nodes drawn from V.

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$.
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
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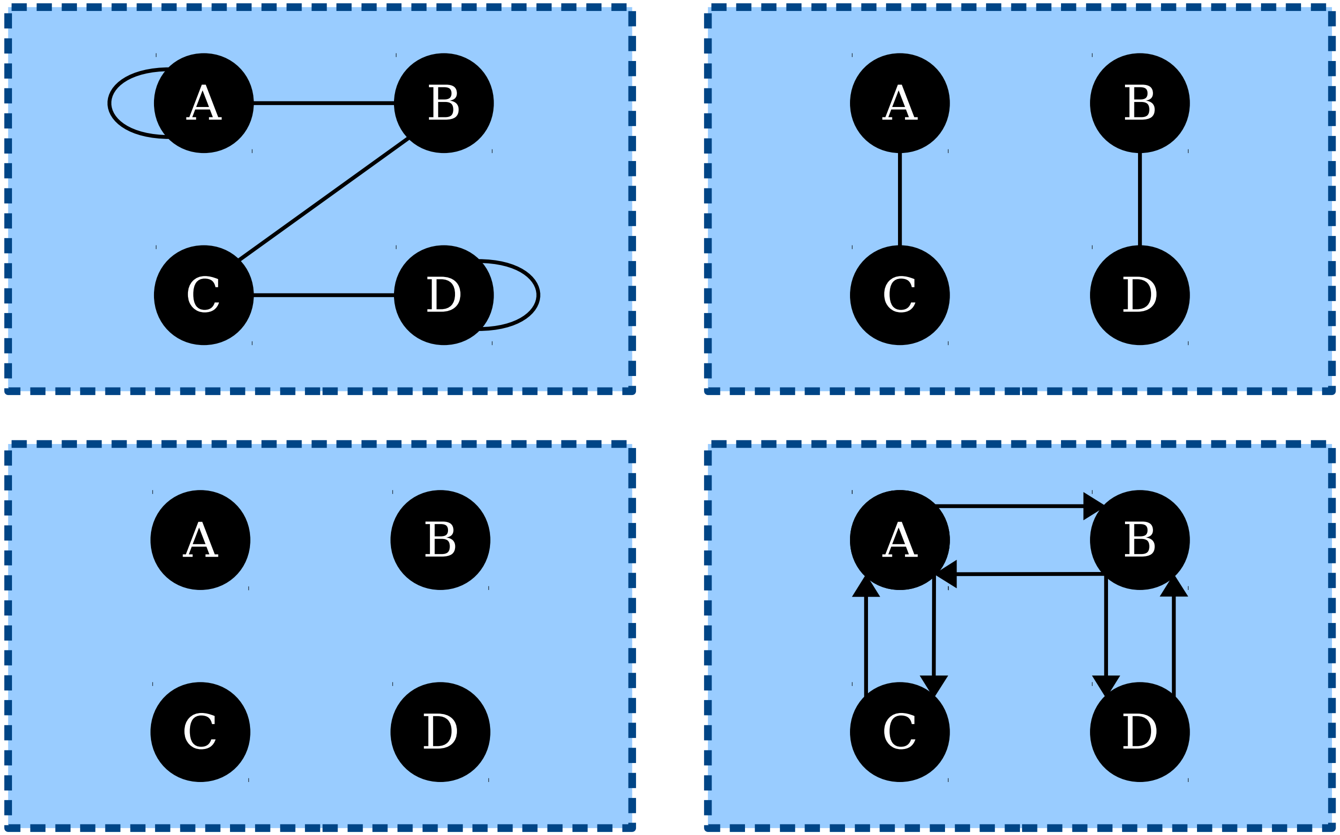
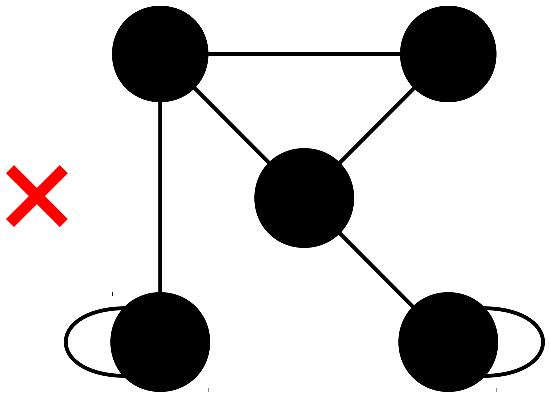
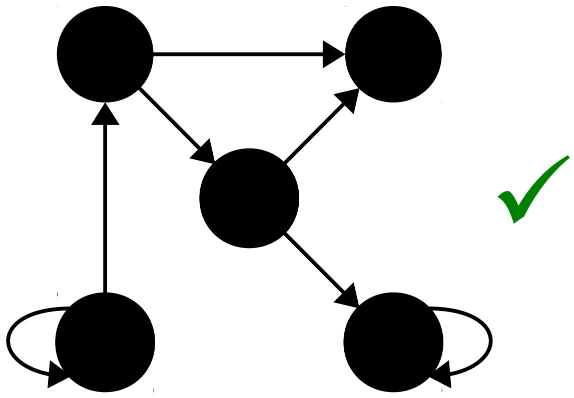


How many of these drawings are of valid undirected graphs?

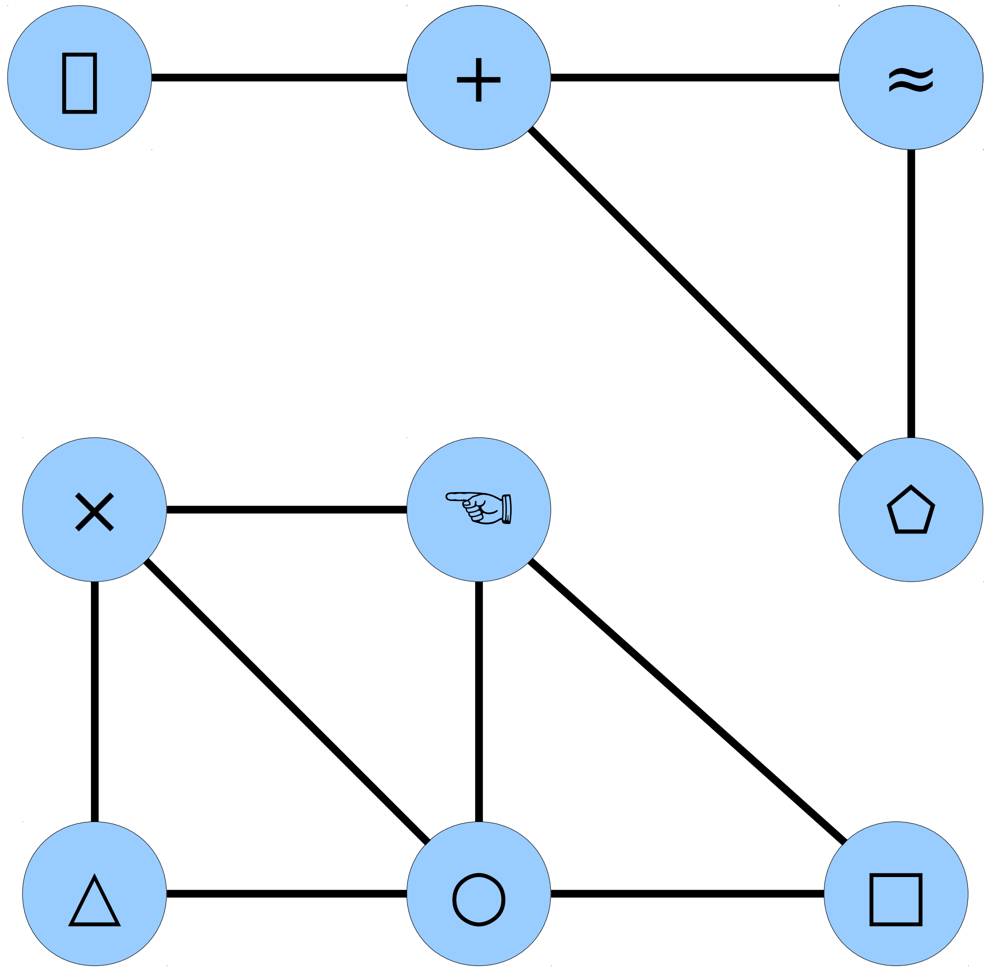
Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
    - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.



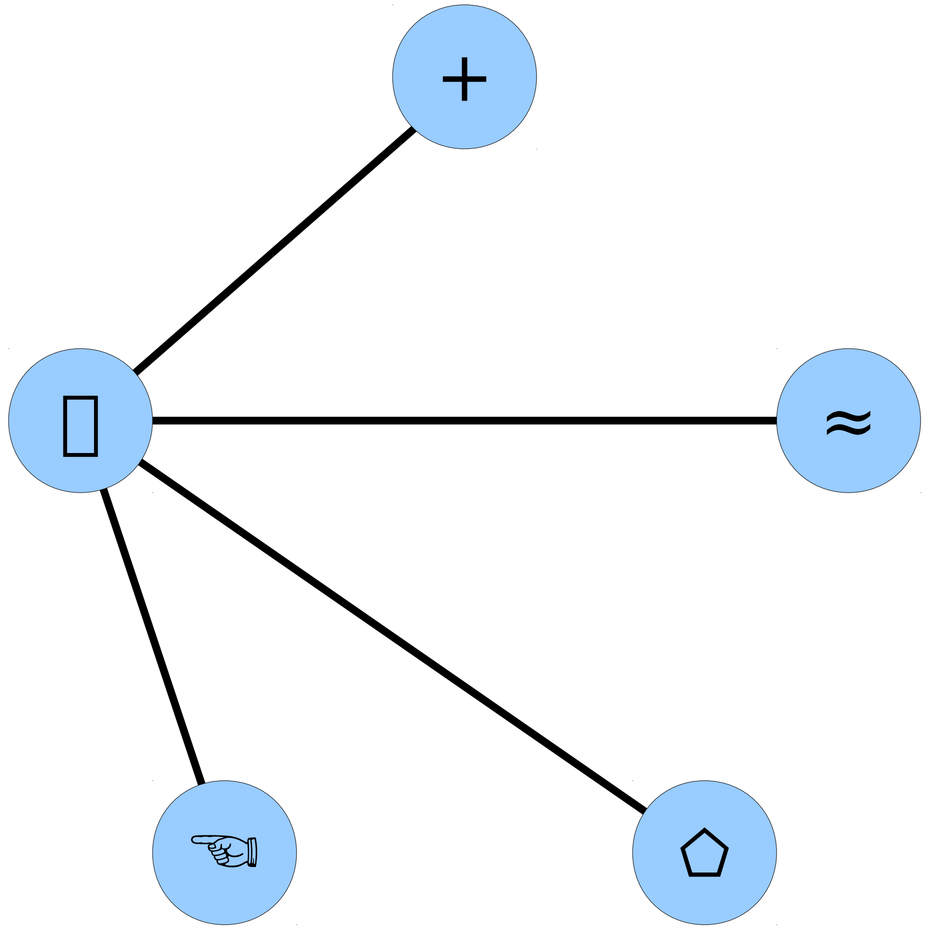


The Great Graph Gallery



Is this formula true about this graph?

$$\forall u \in V.\ \exists v \in V.\ \{u, v\} \in E$$



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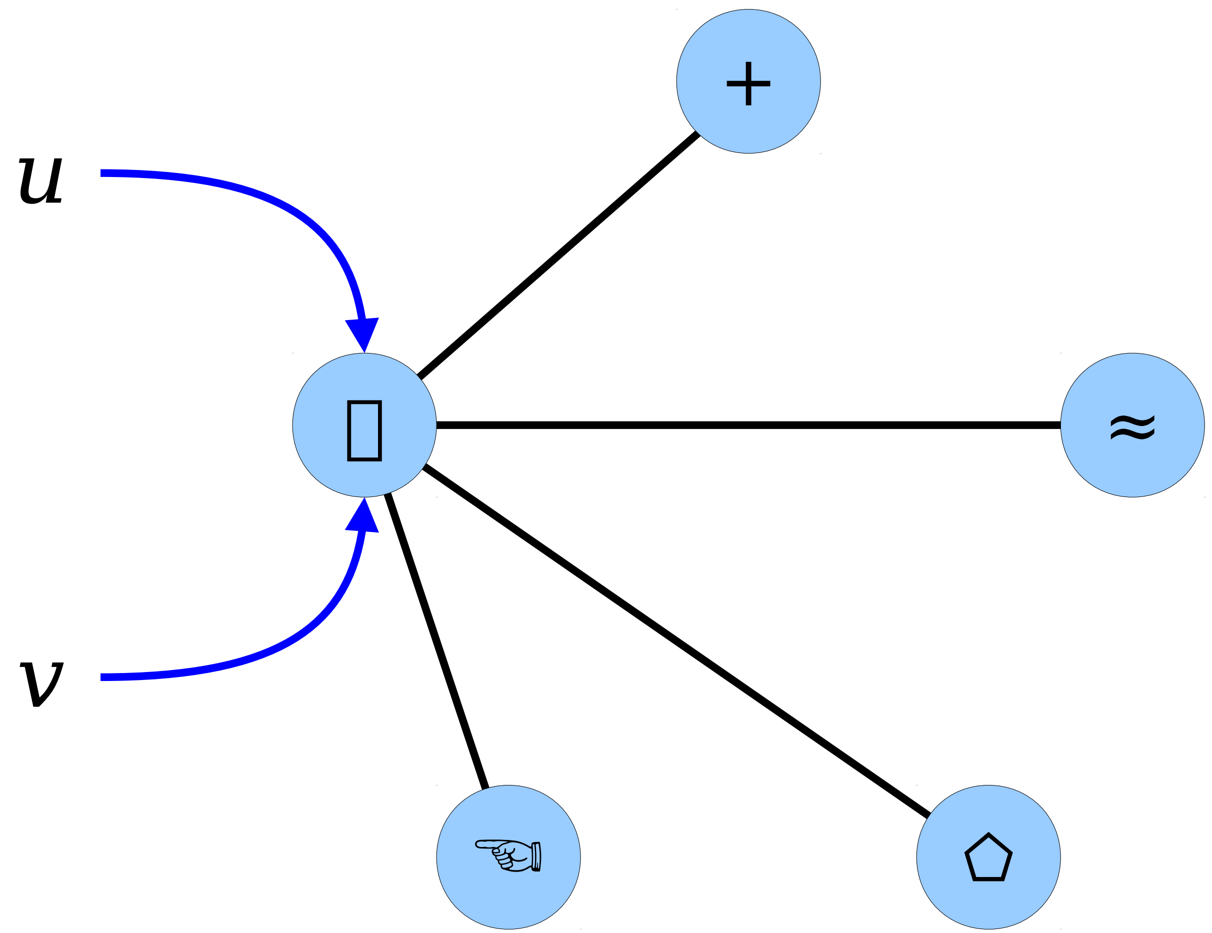

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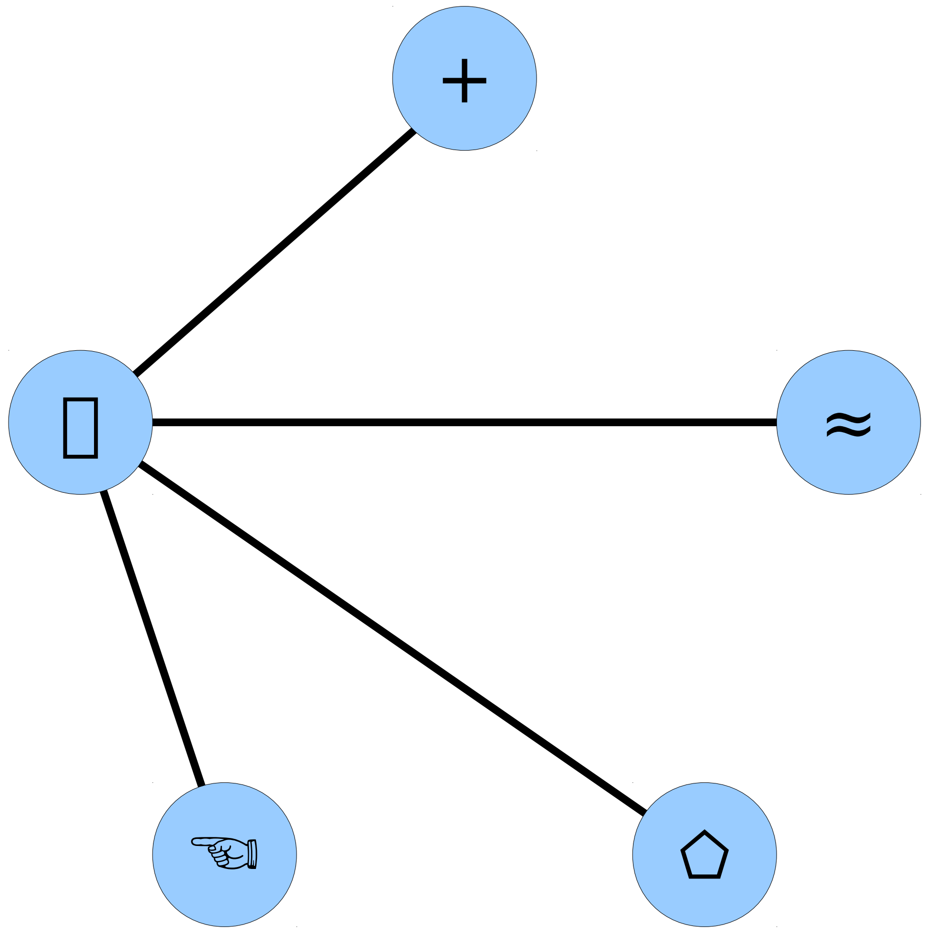
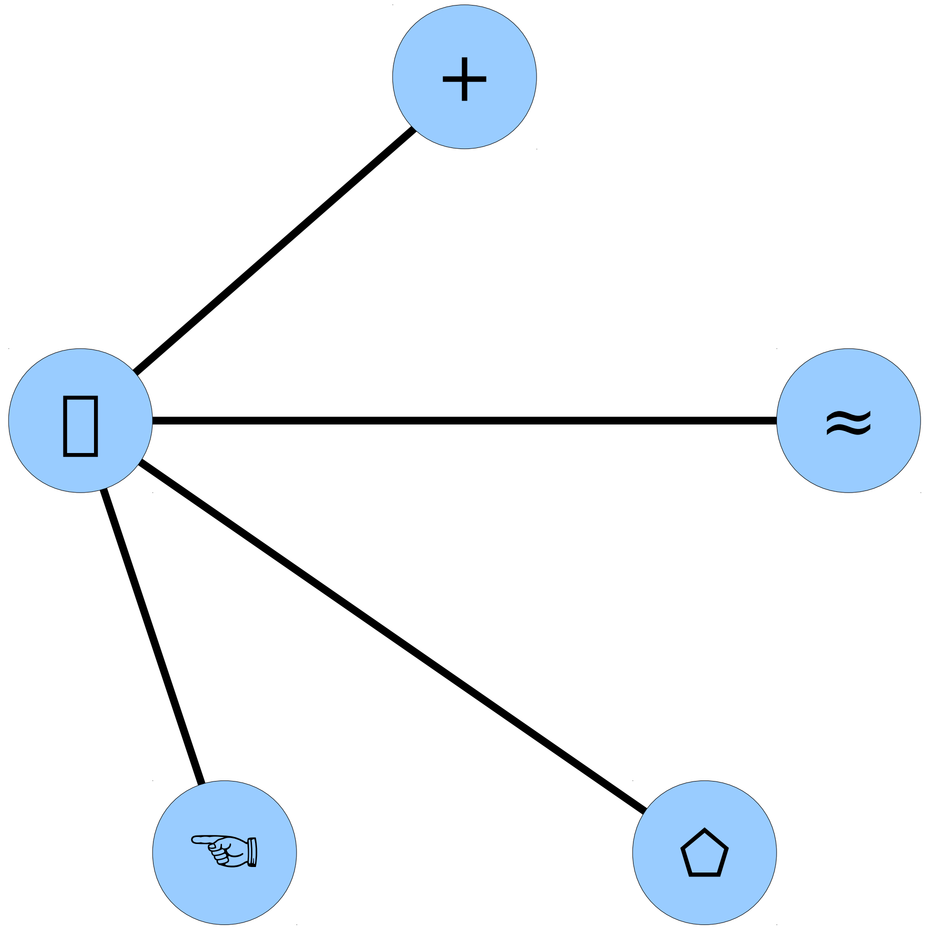


Let's look at the negation!

$$\exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



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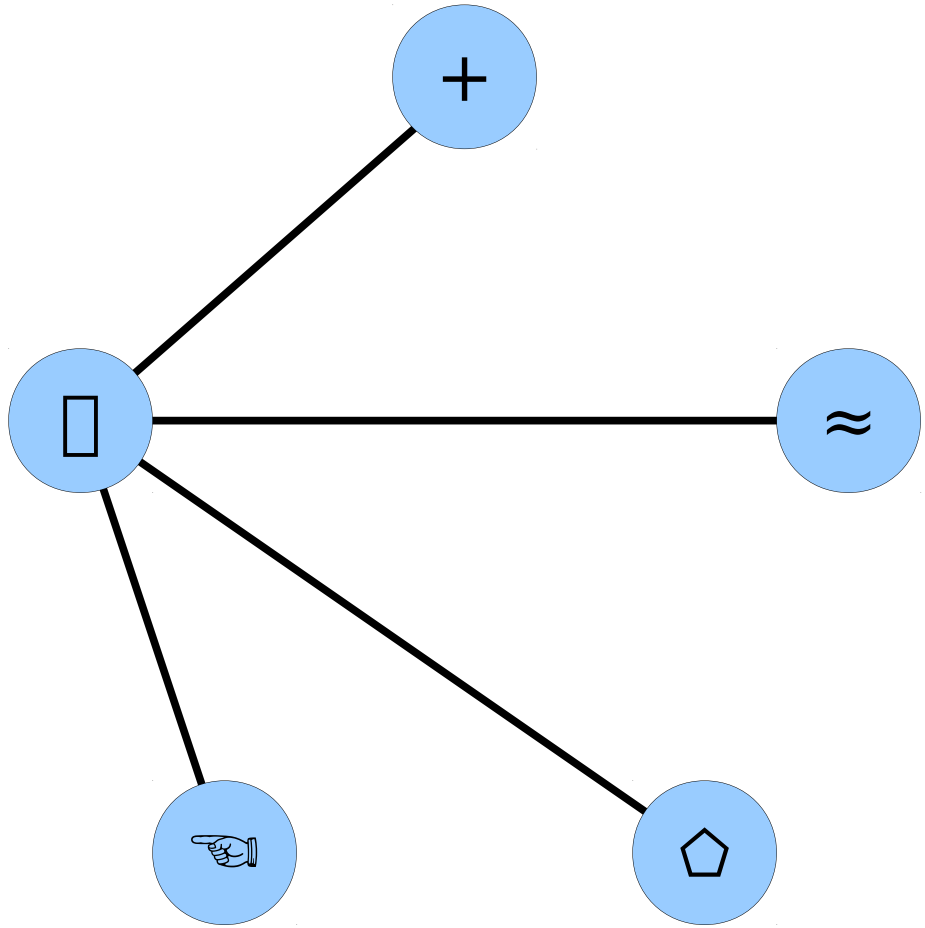

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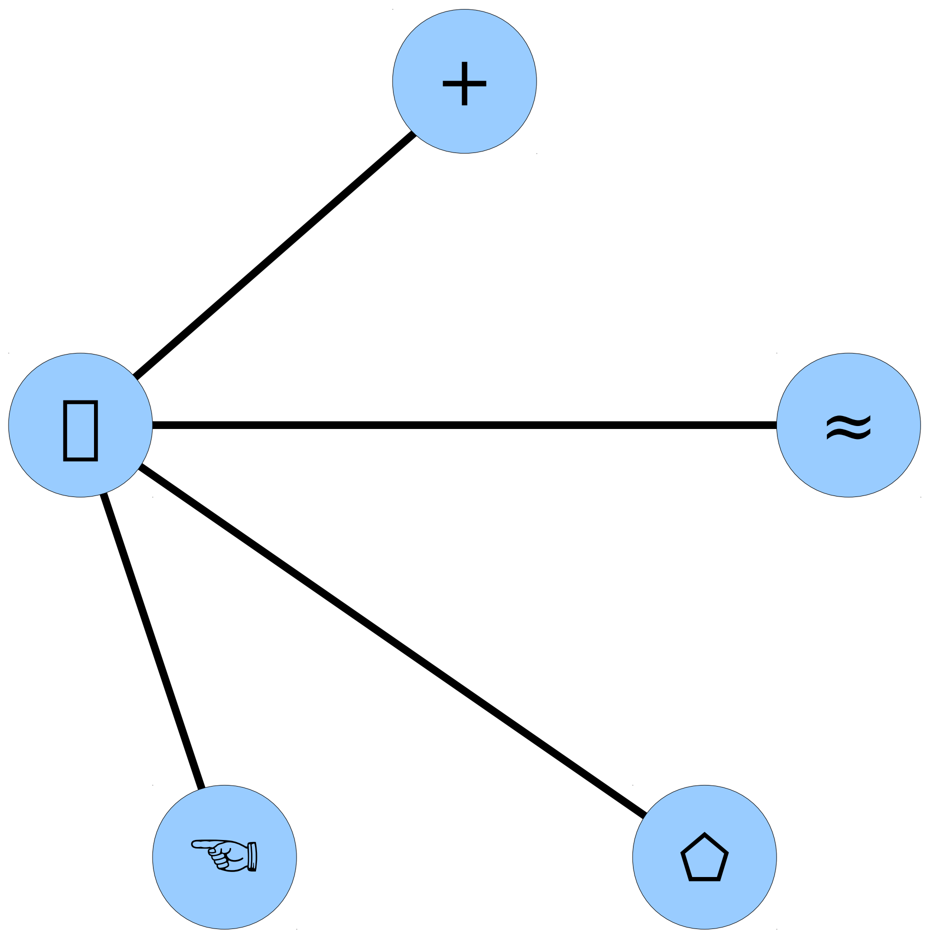

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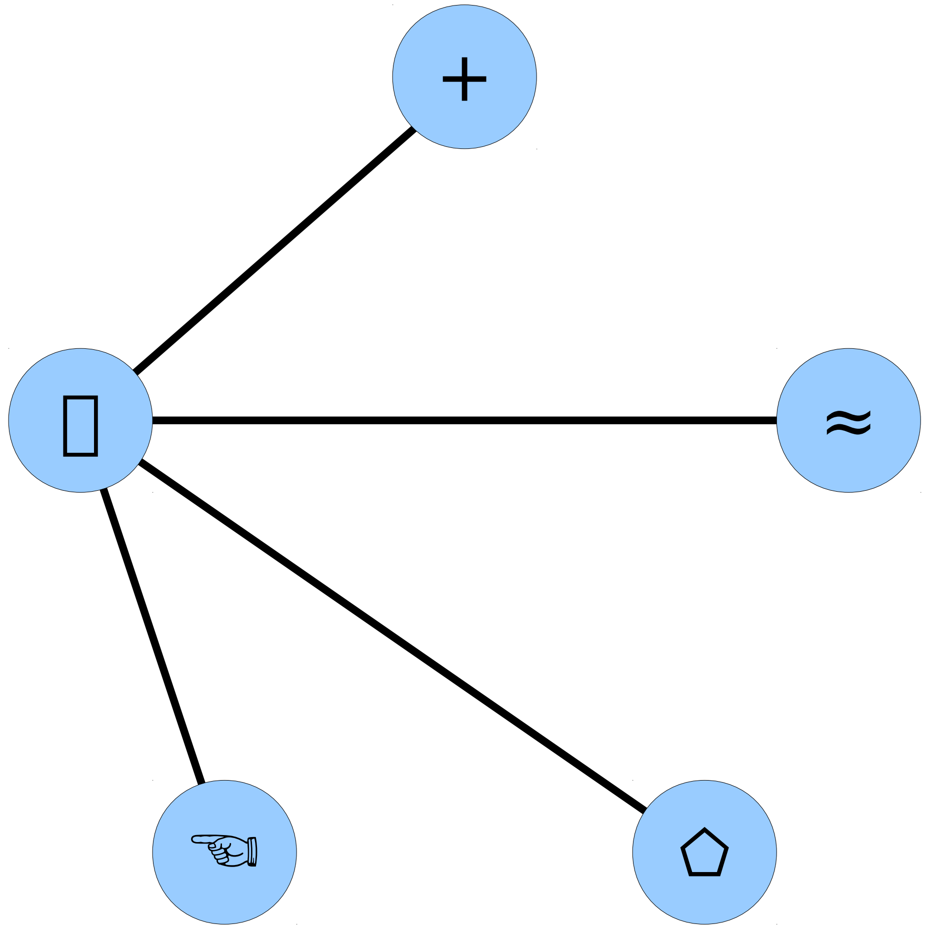
$$\forall u \in V.\ \exists v \in V.\ \neg(\{u, v\} \in E)$$



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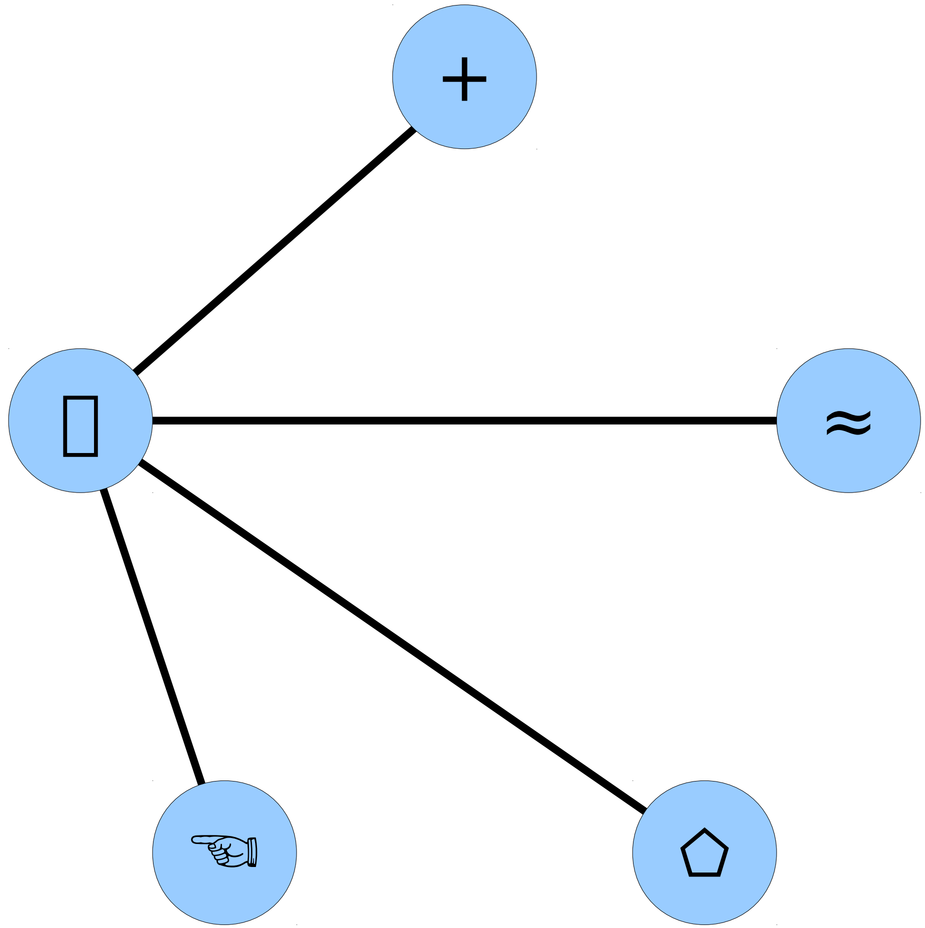

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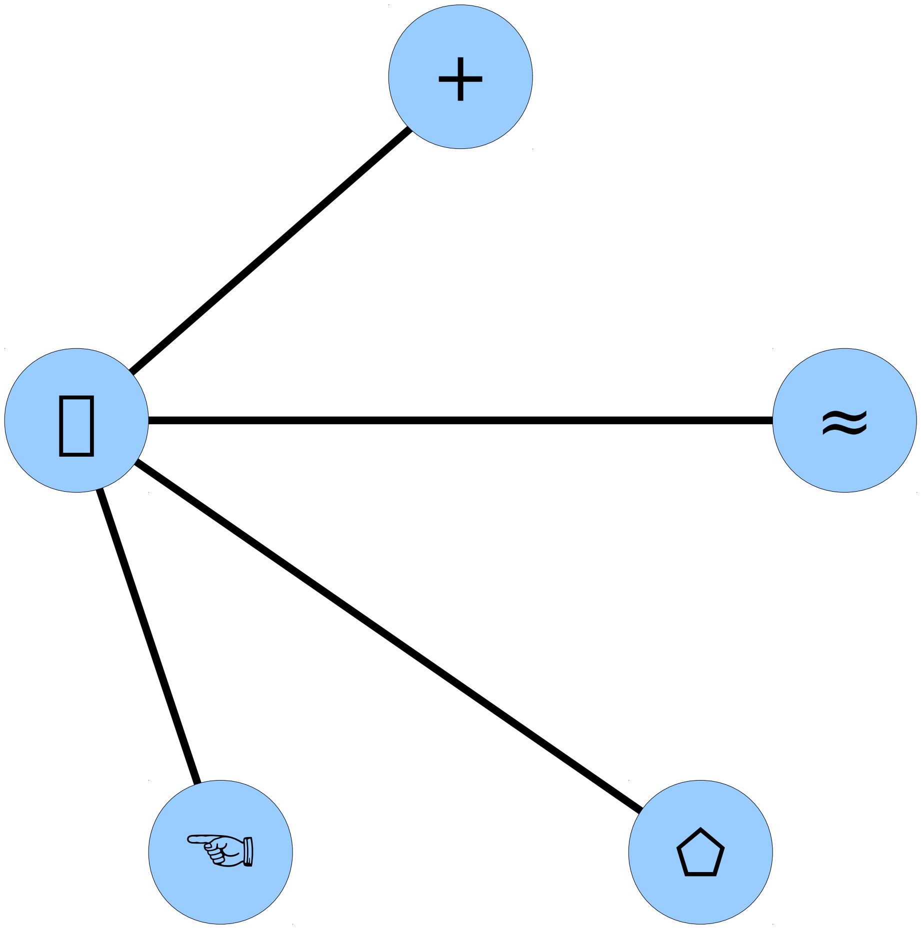

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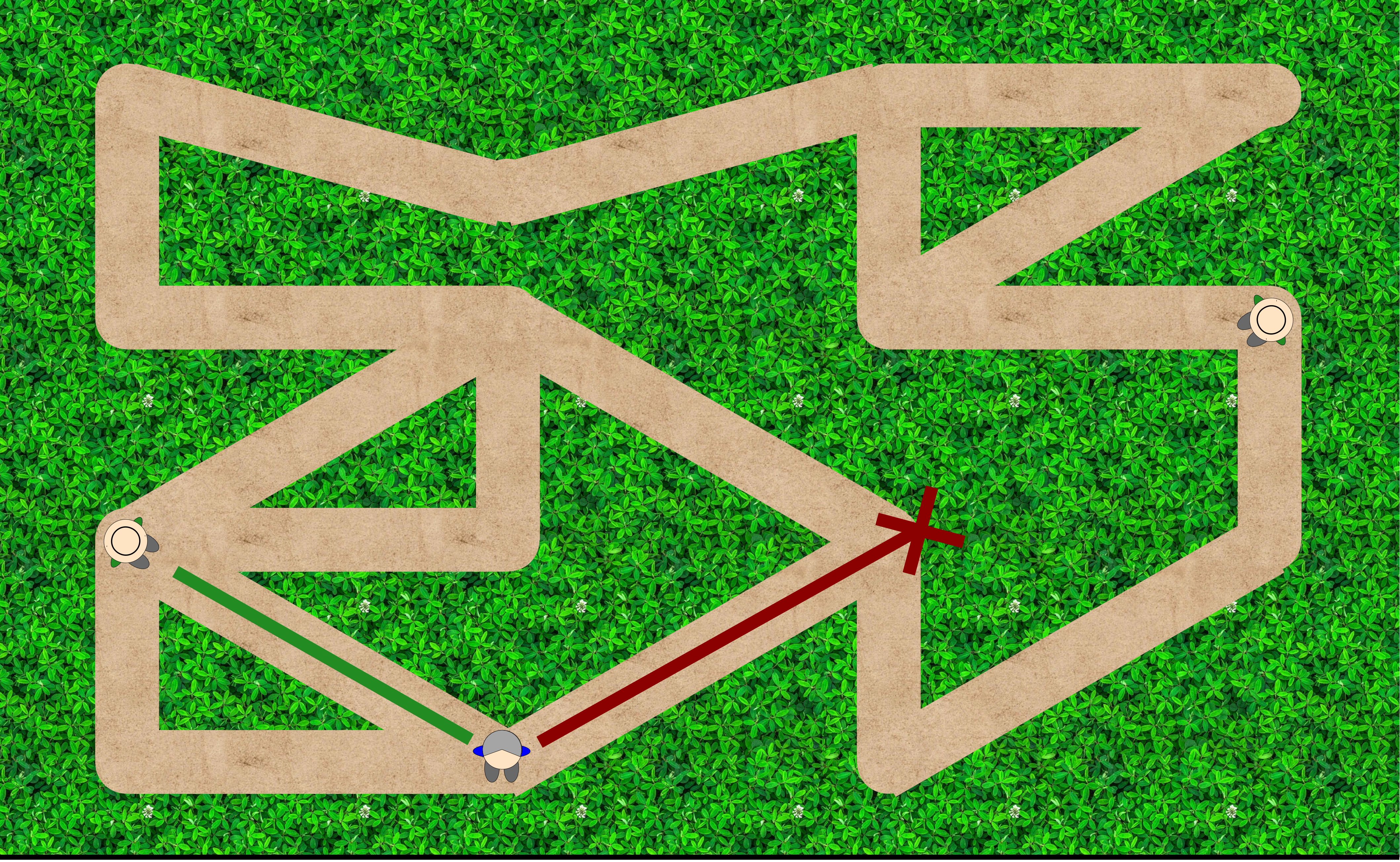
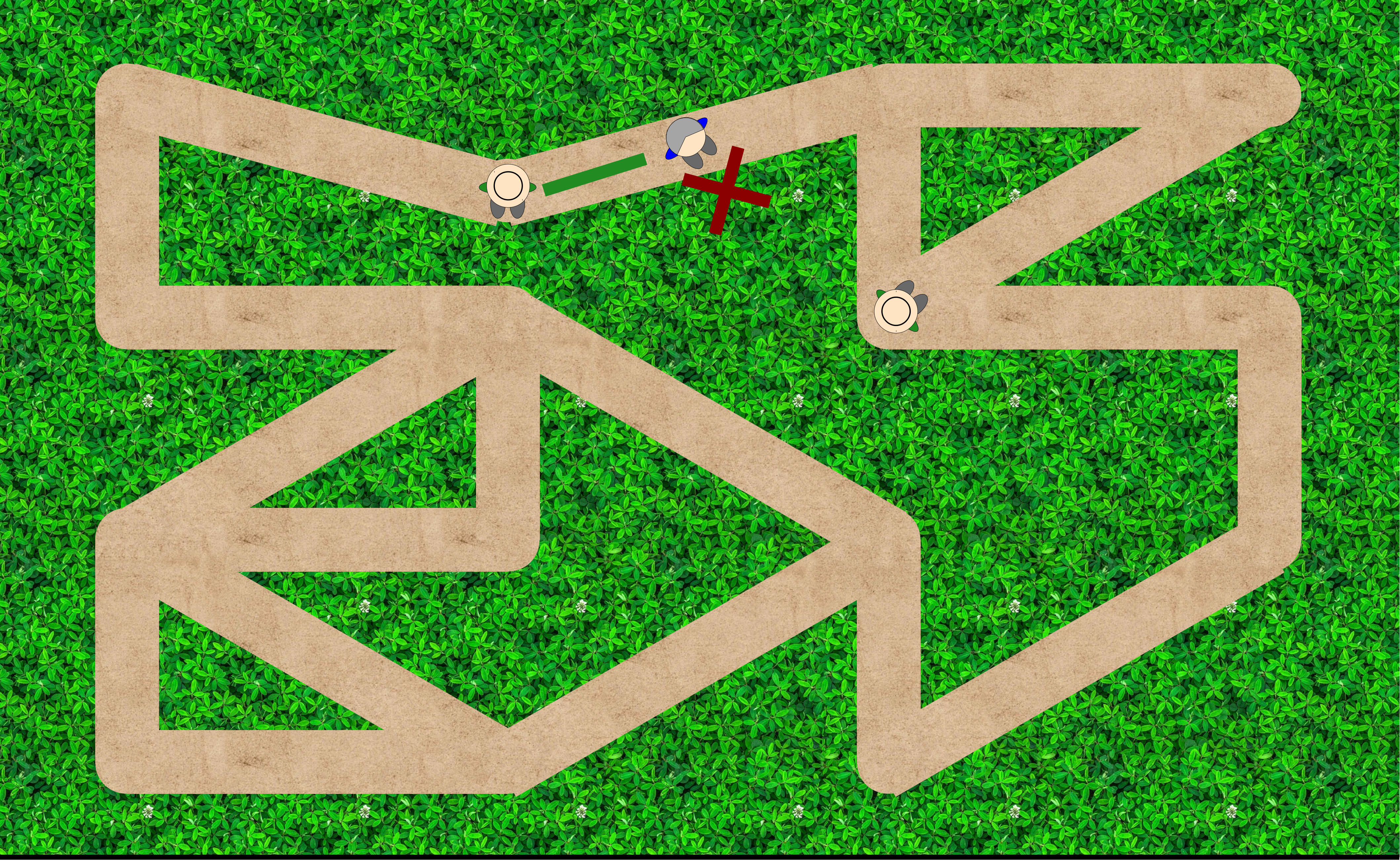
Independent Sets and Vertex Covers

Two Motivating Problems



Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



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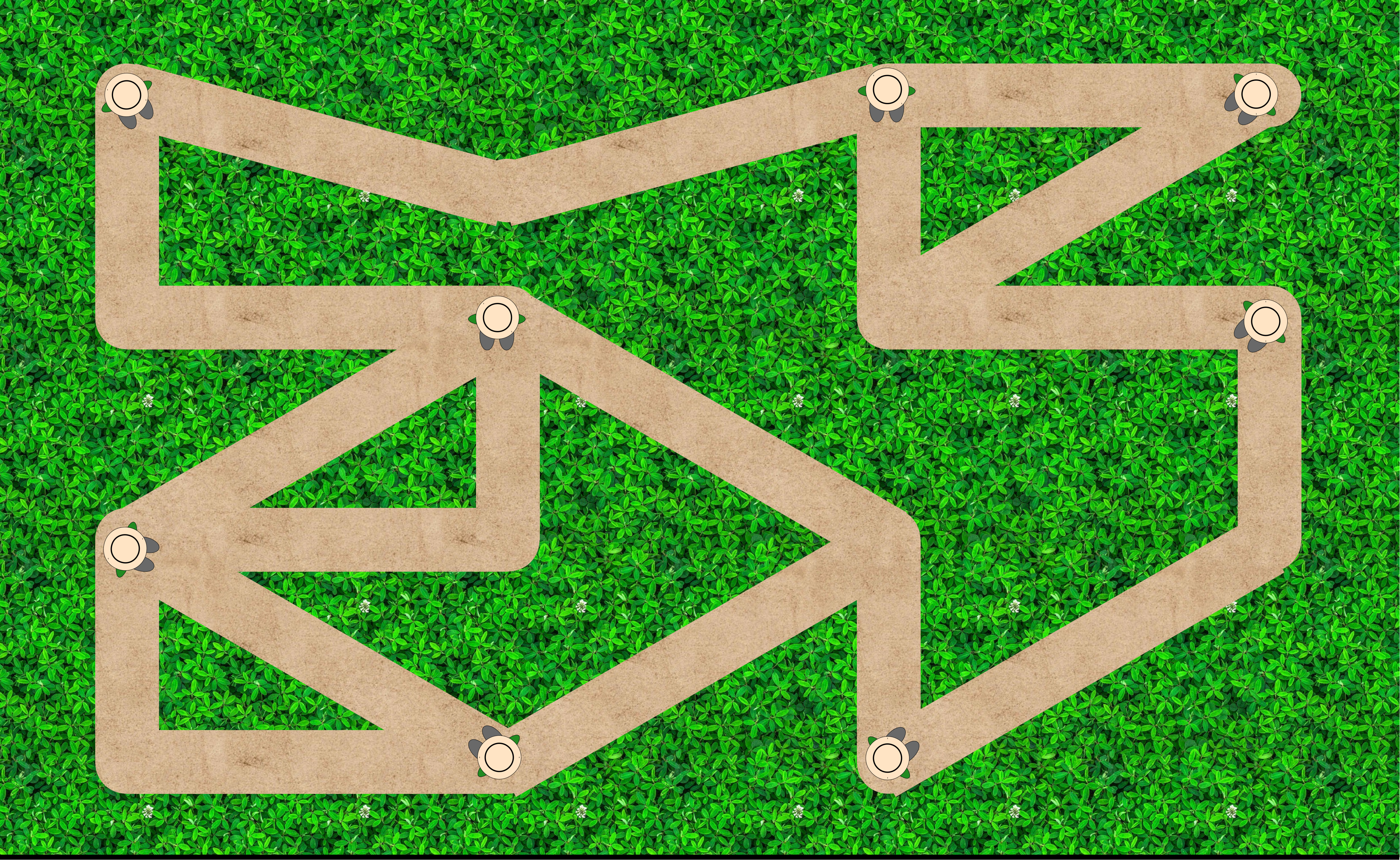

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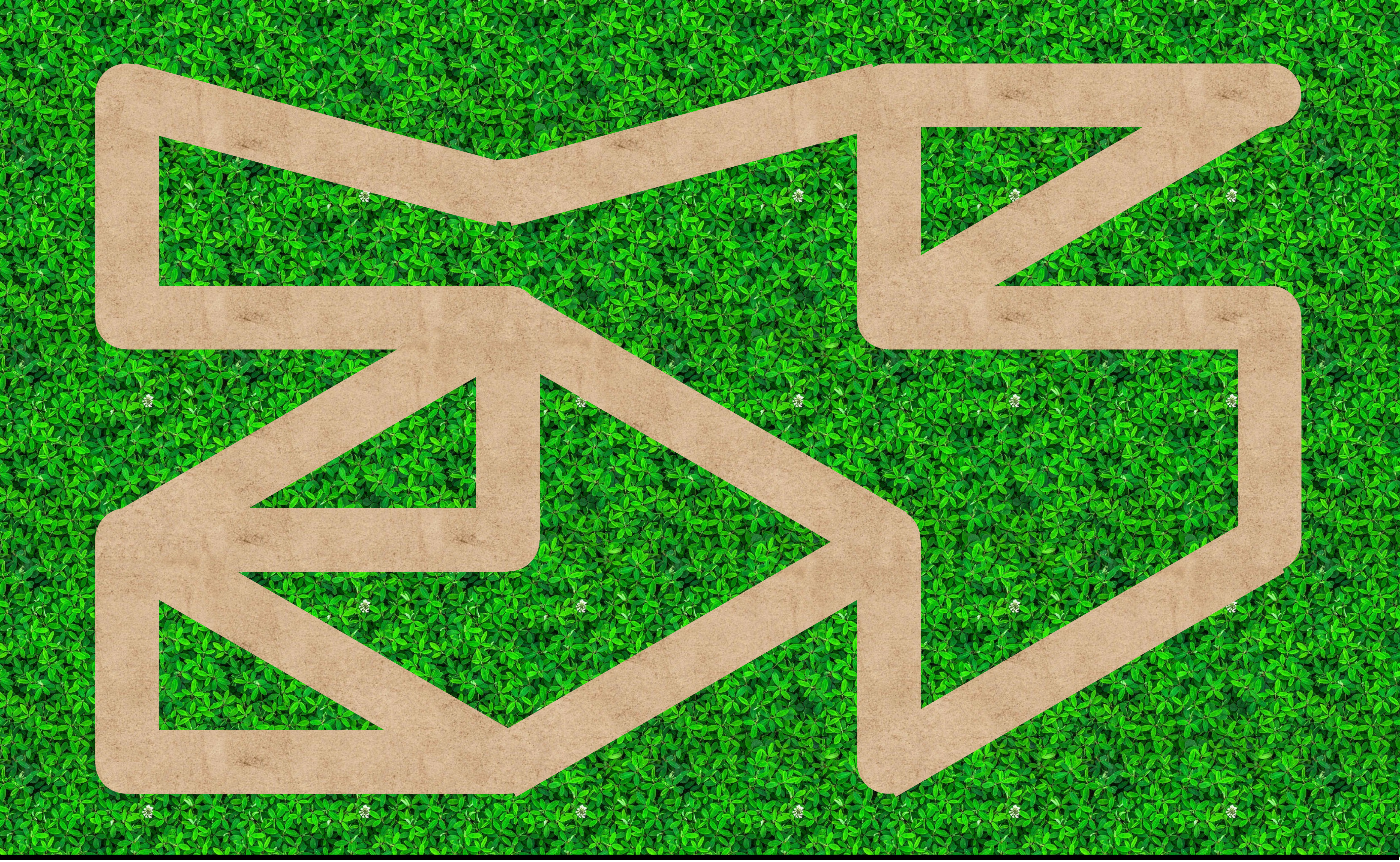

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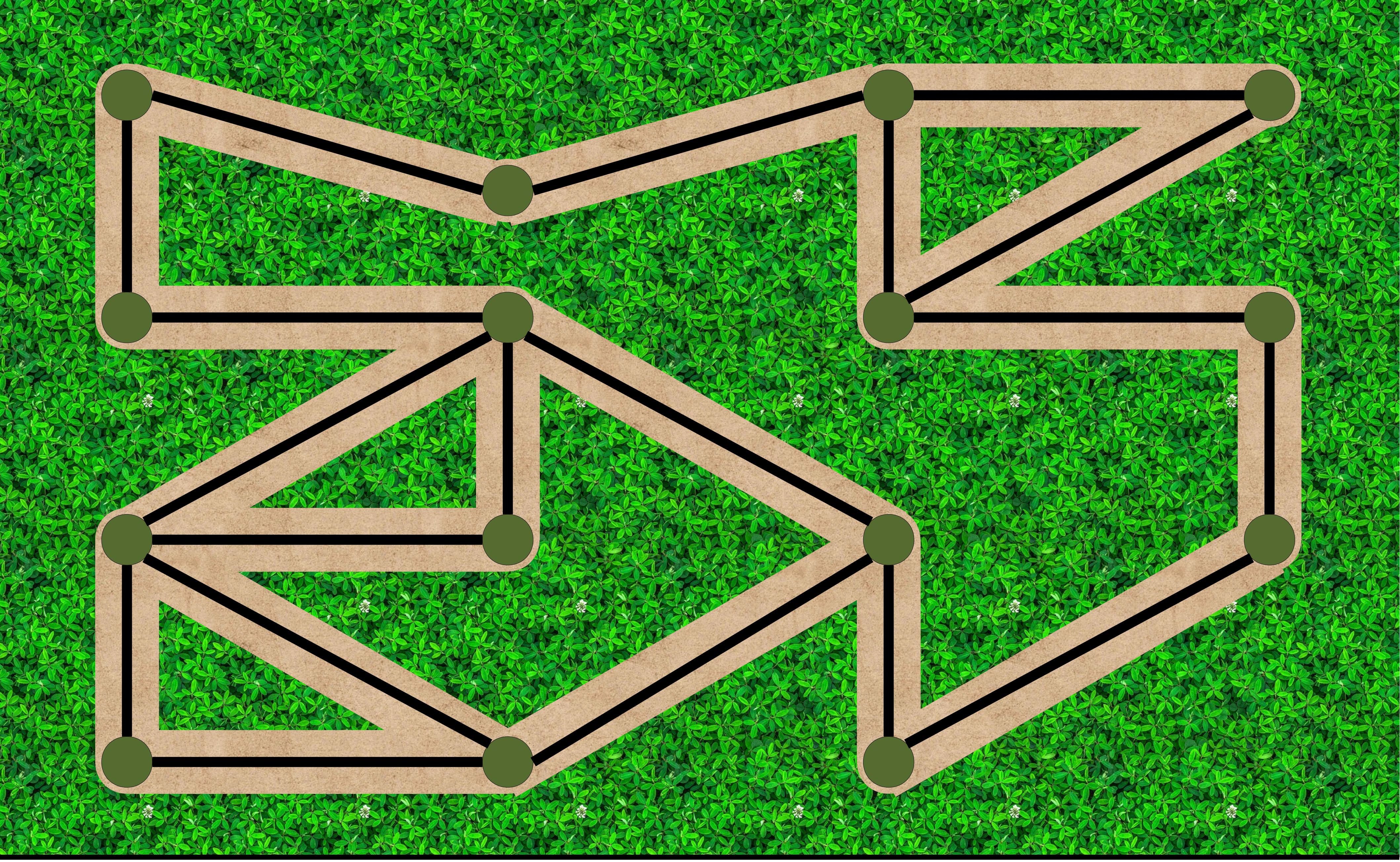

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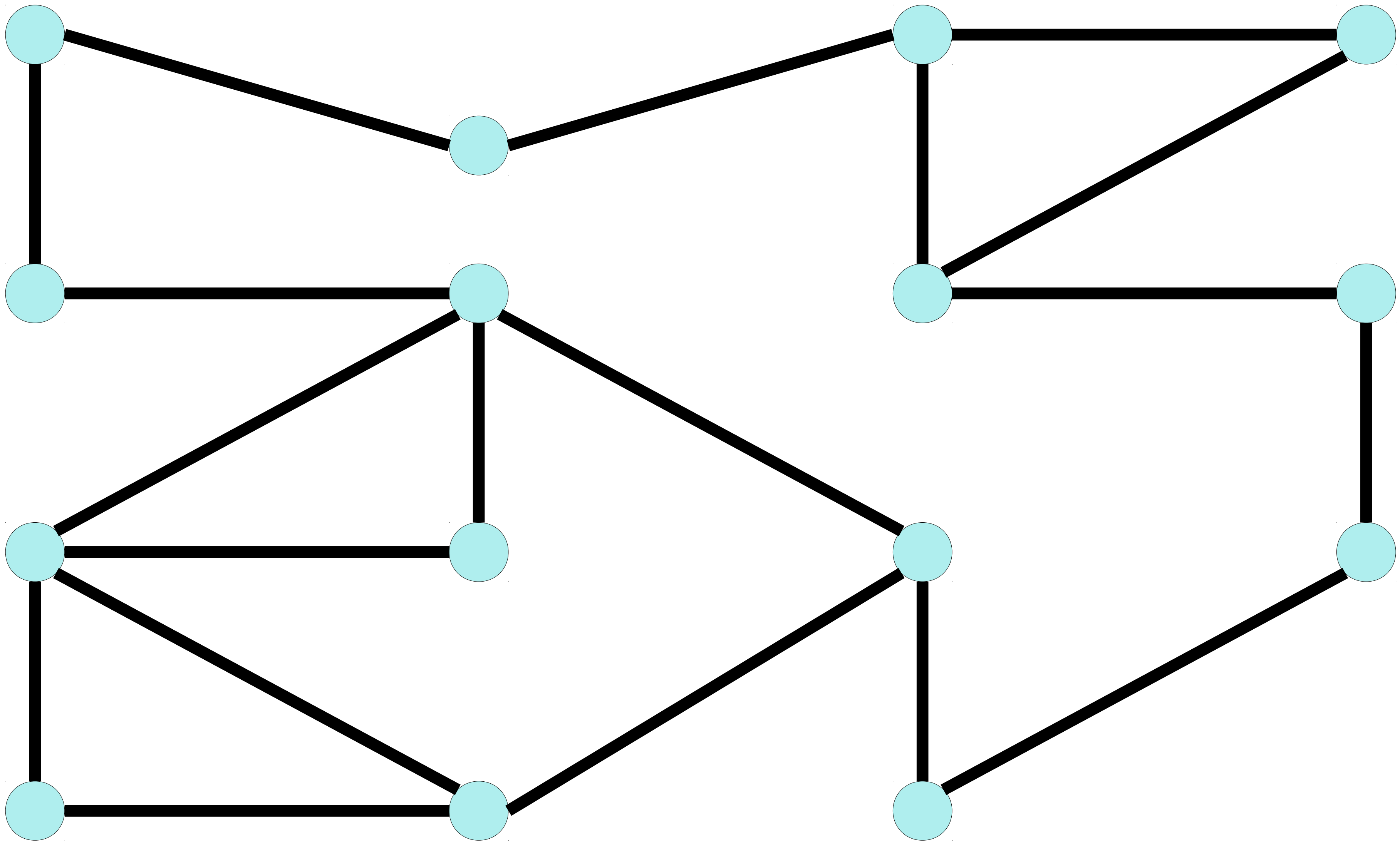

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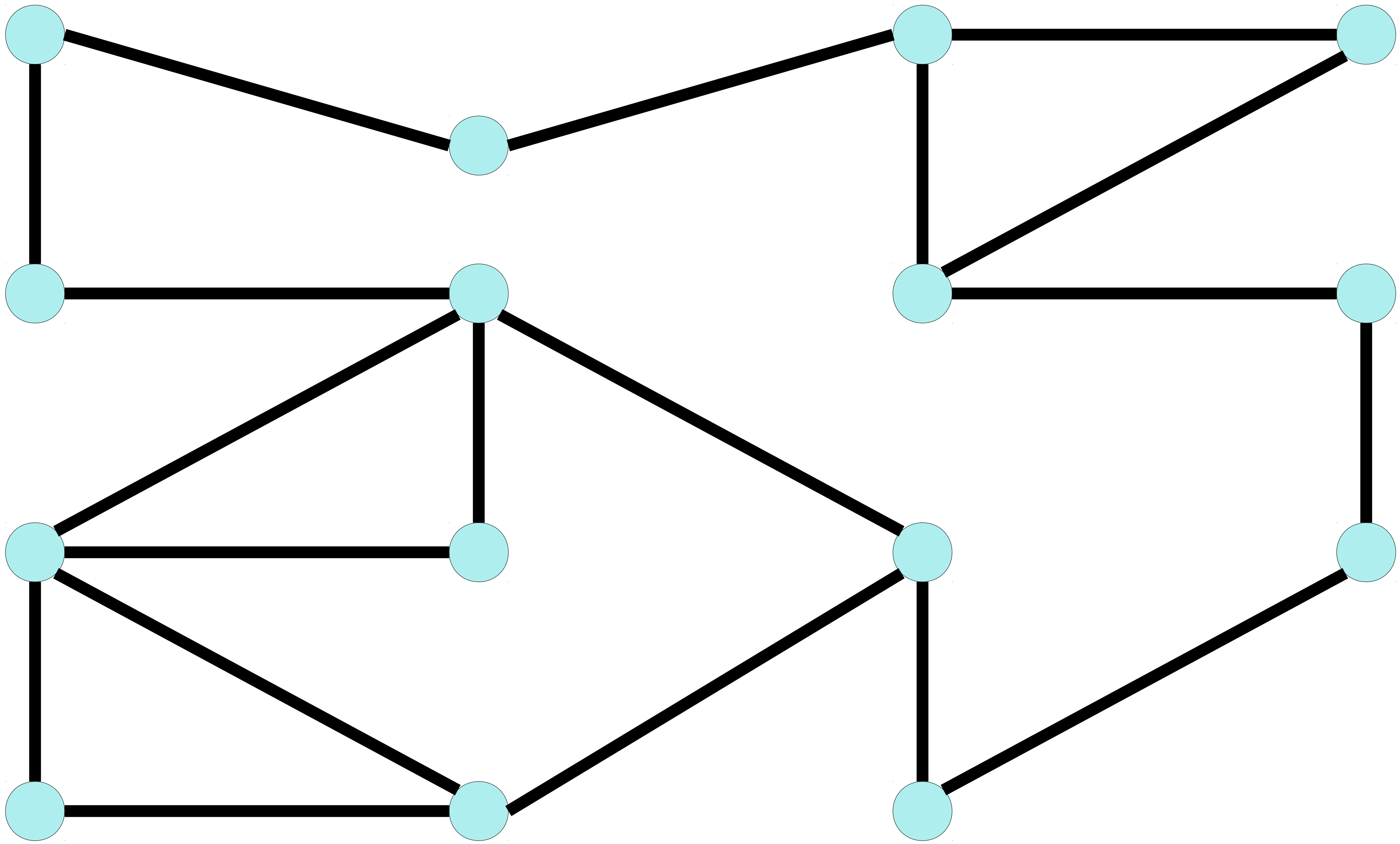
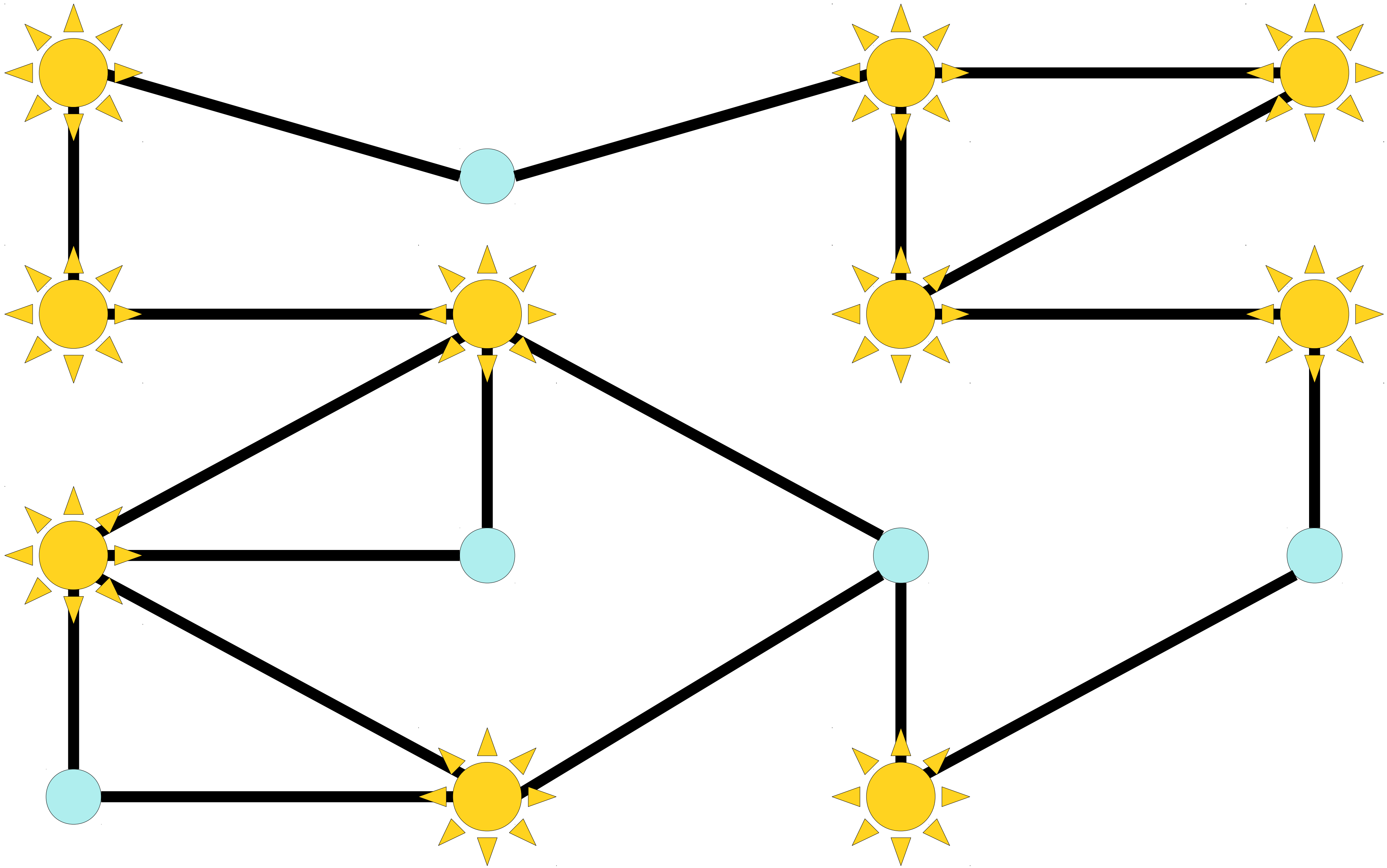


Choose at least one endpoint of each edge.



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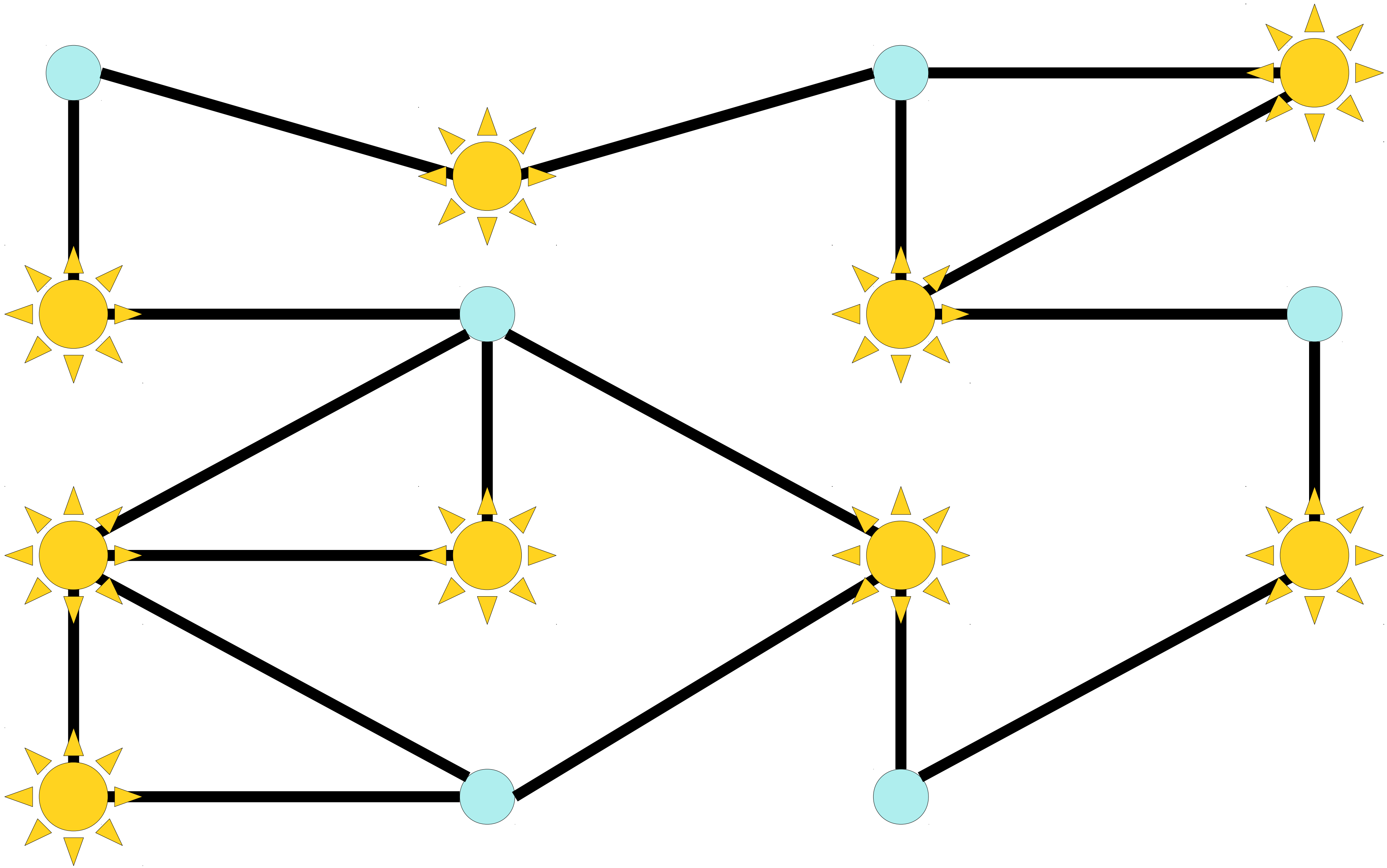

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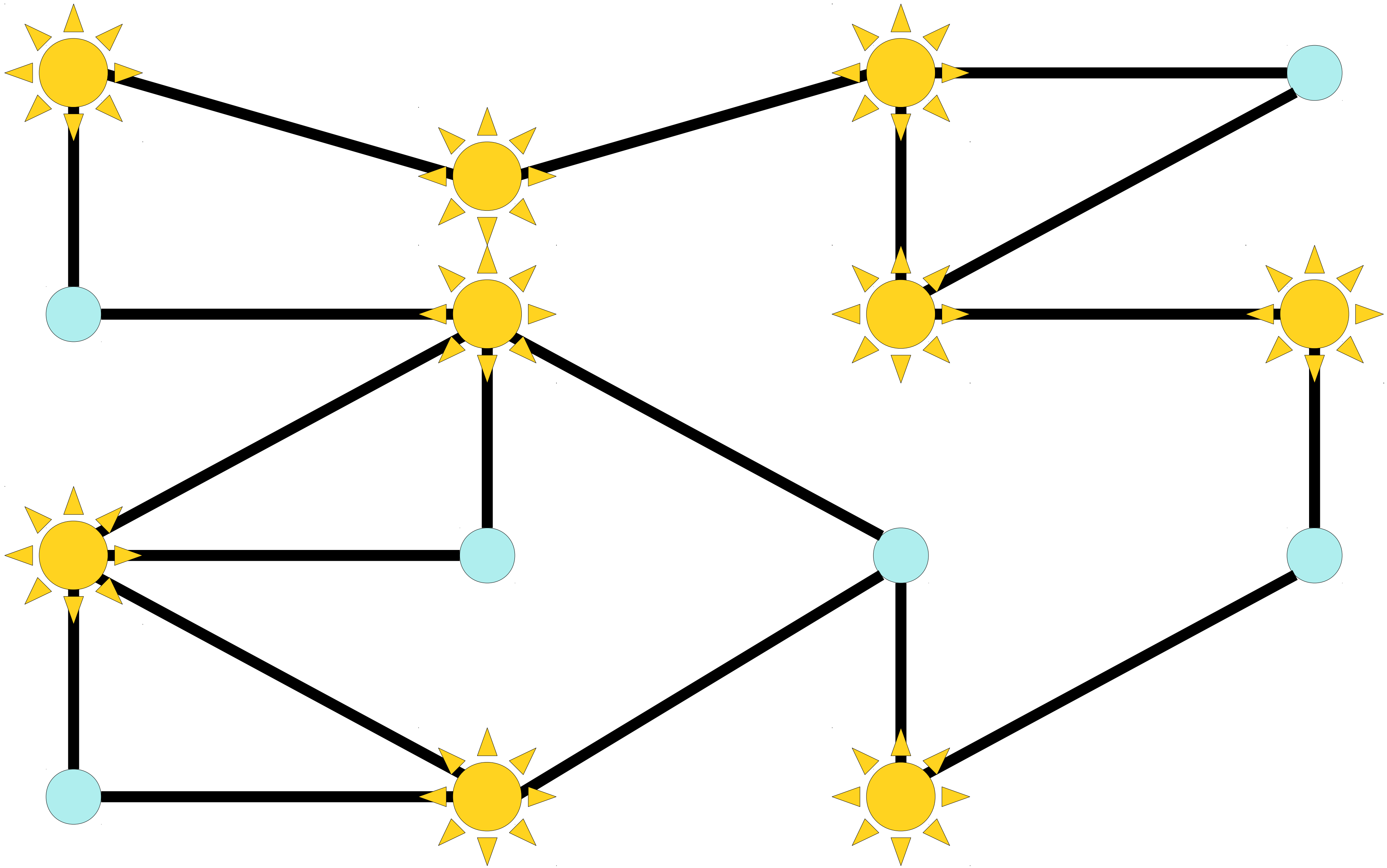


Choose at least one endpoint of each edge.

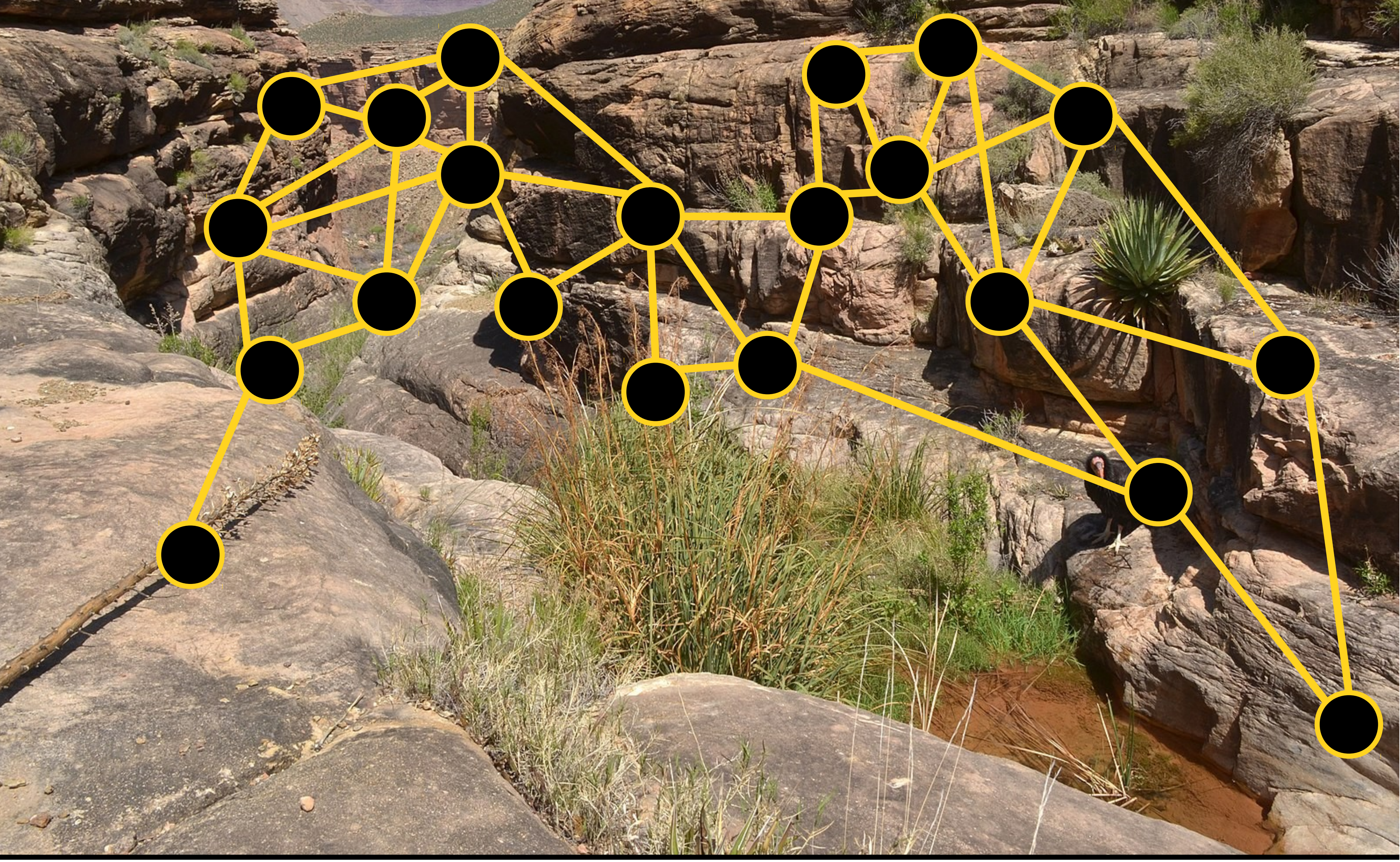
Vertex Covers

- Let $G = (V, E)$ be an undirected graph. A ***vertex cover*** of G is a set $C \subseteq V$ such that the following statement is true:

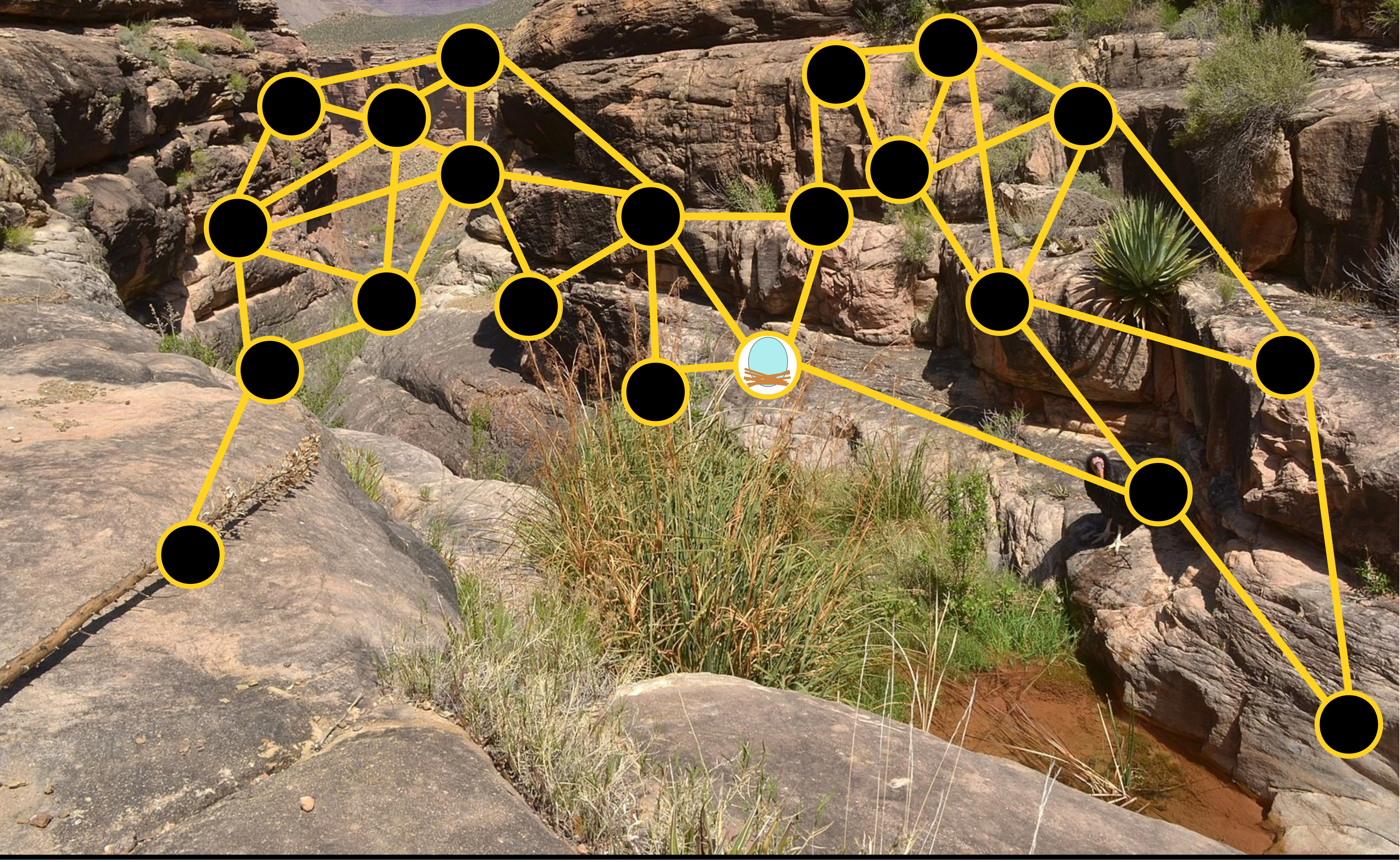
$$\forall x \in V.\ \forall y \in V.\ (\{x, y\} \in E \to (x \in C \lor y \in C))$$

("Every edge has at least one endpoint in C.")

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.



Set up nests for the California condor. Condors are territorial and won't nest if they can see other condors.



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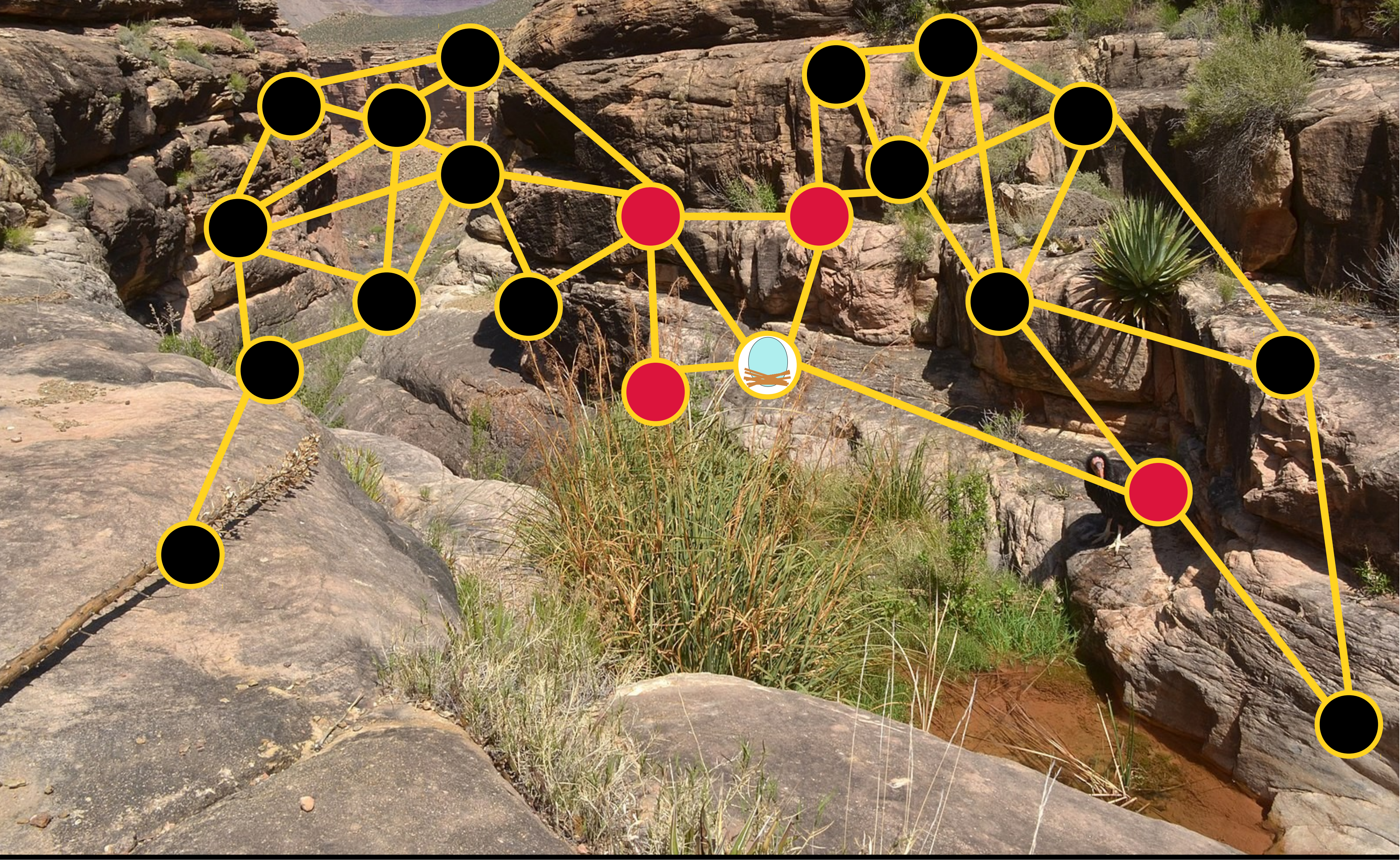

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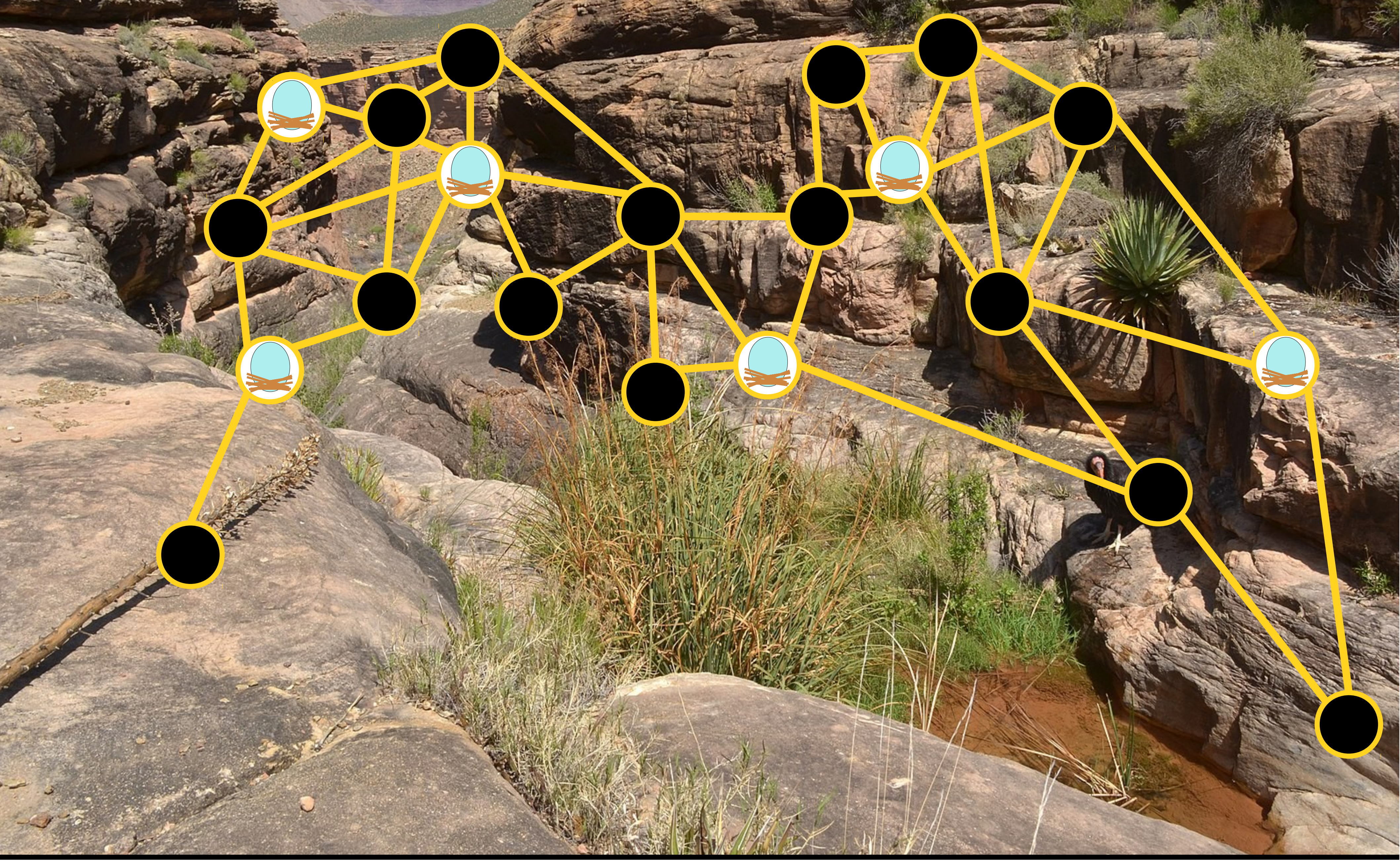

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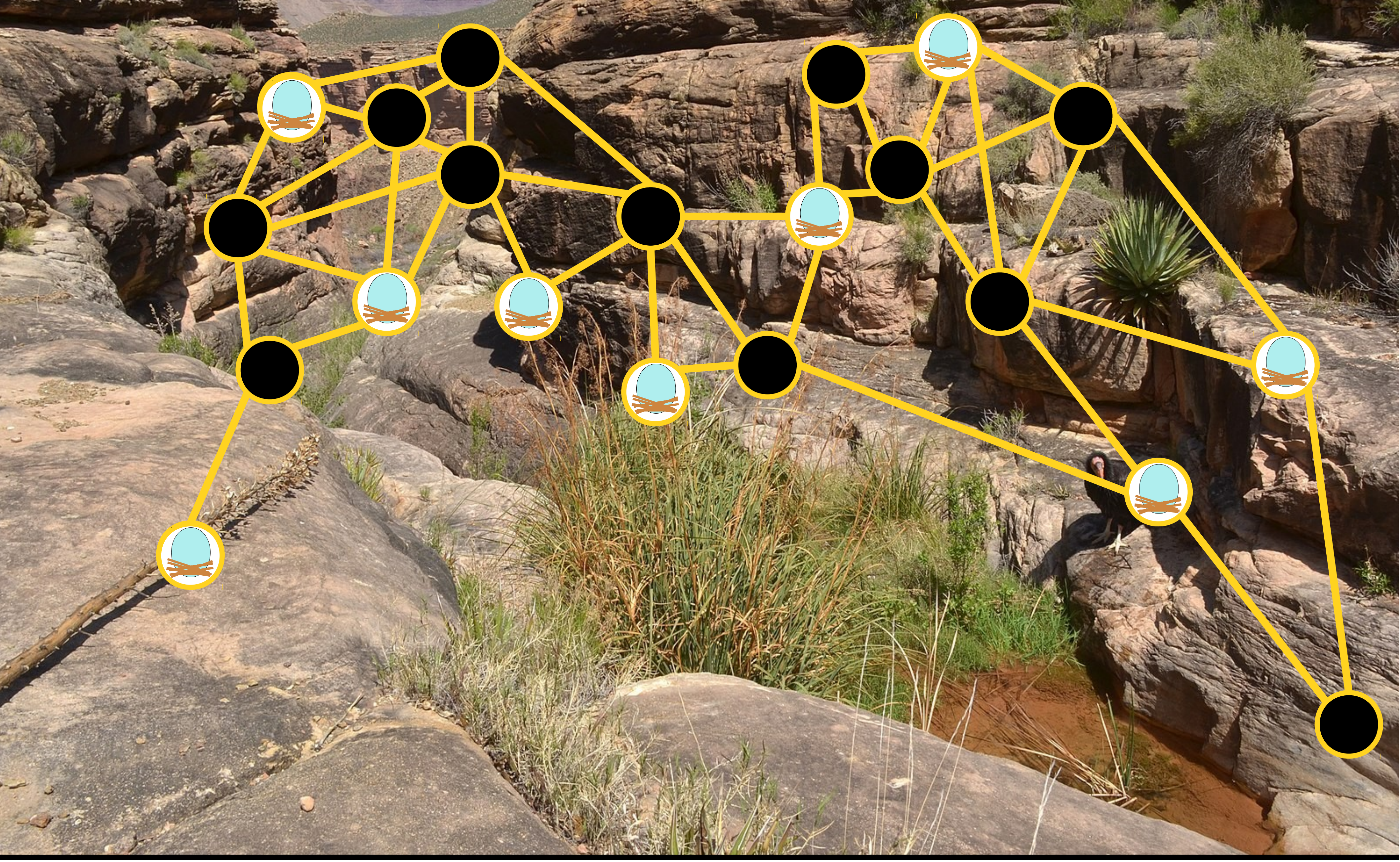

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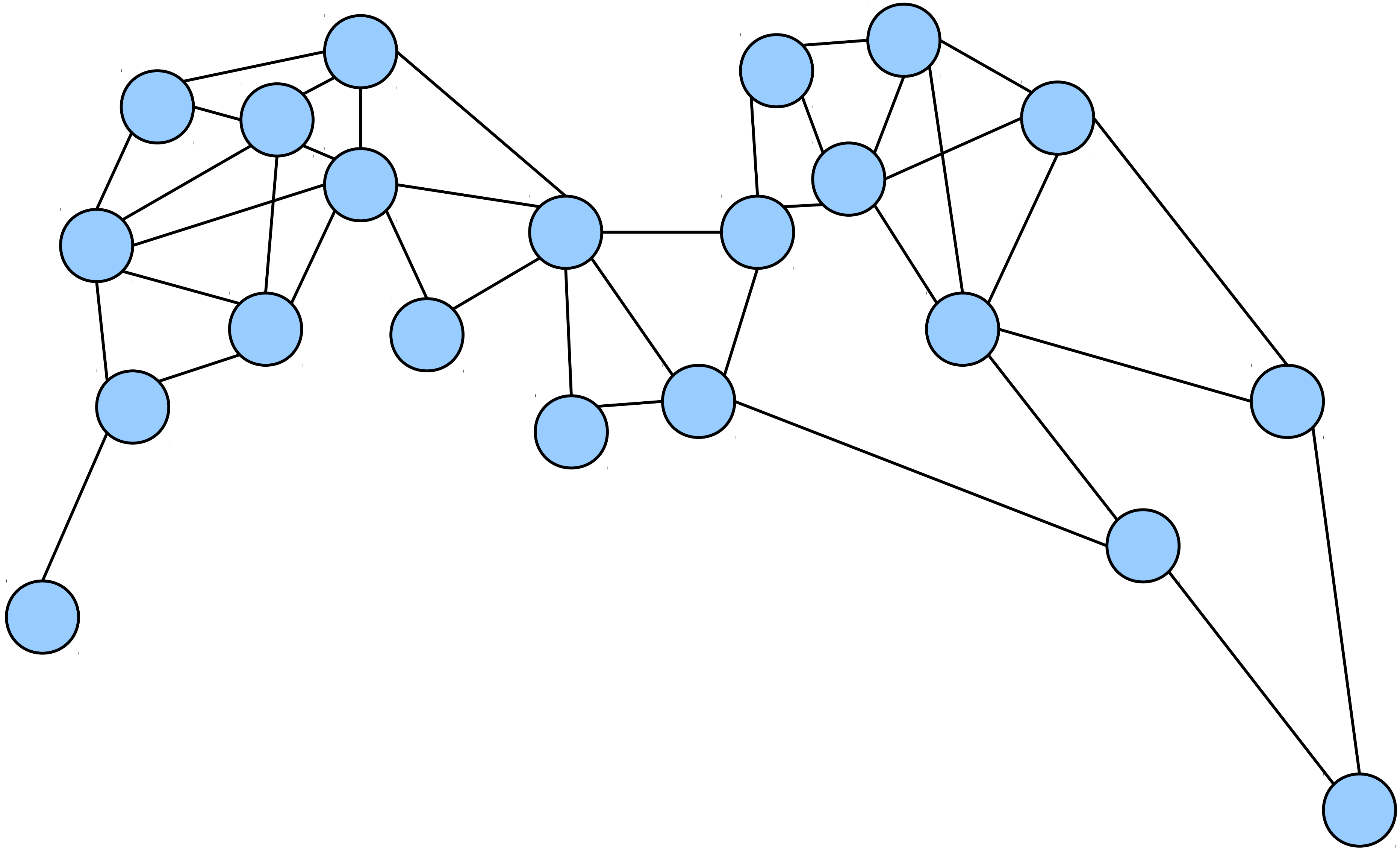

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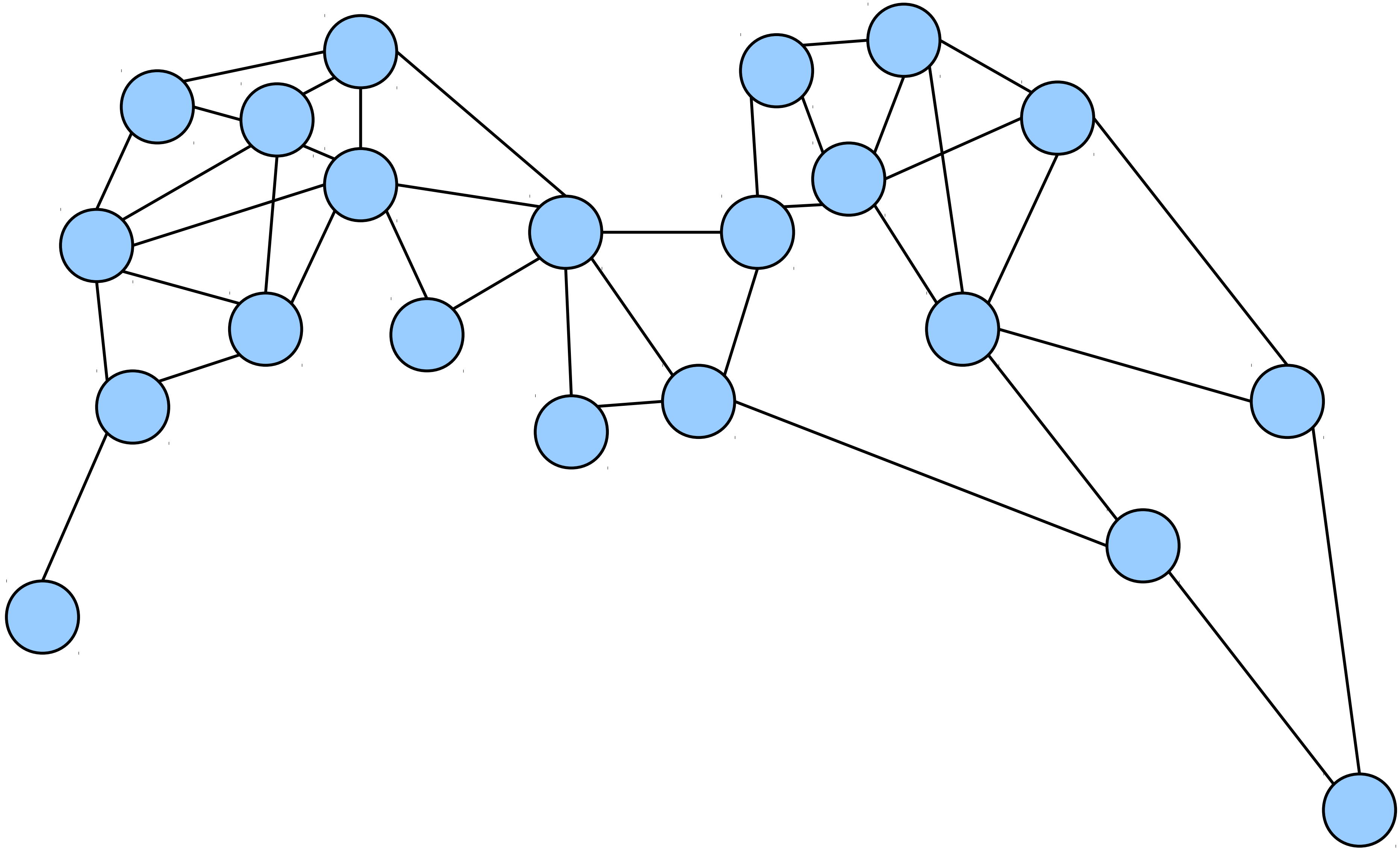


Choose a set of nodes, no two of which are adjacent.

Independent Sets

- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

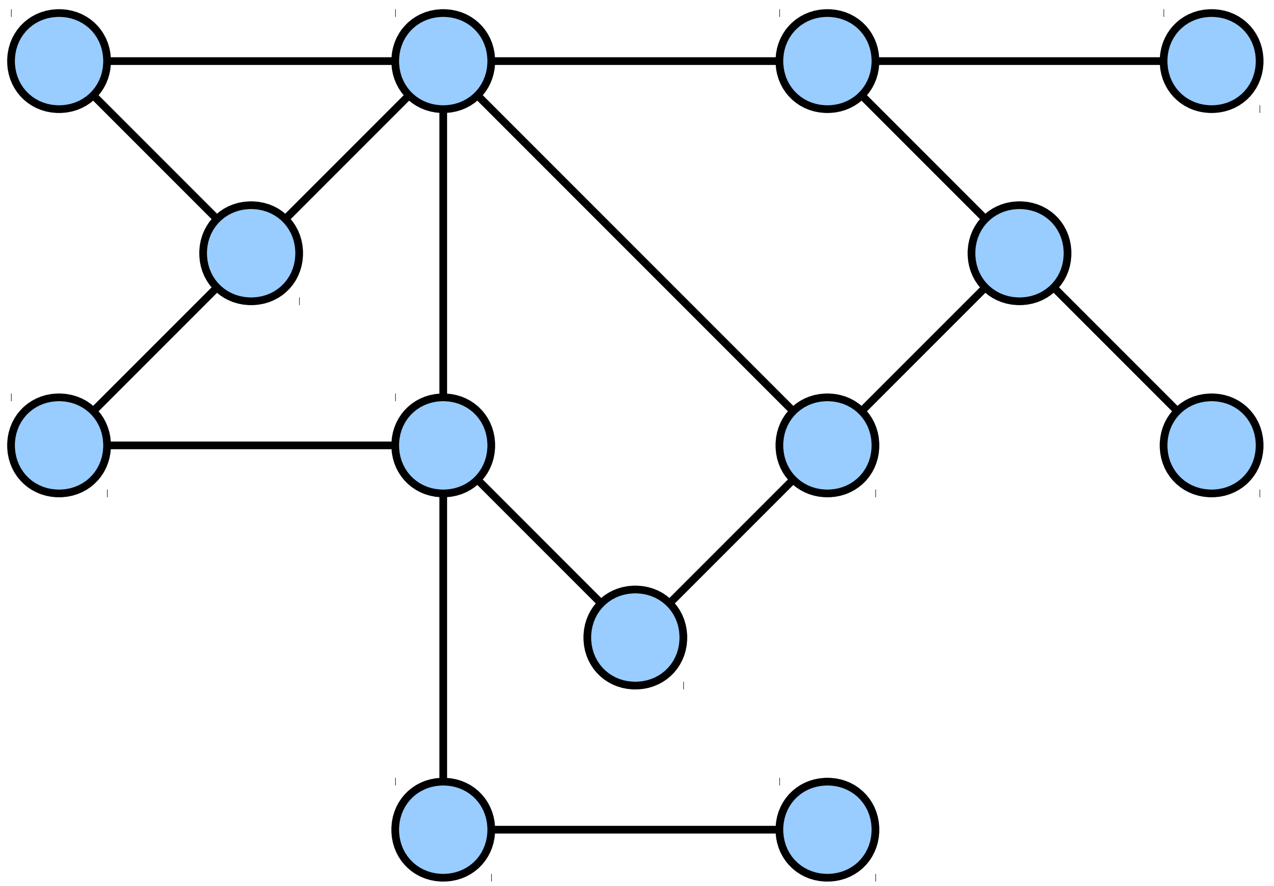
$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

("No two nodes in I are adjacent.")

- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.

Constraint Optimization with Independent Set and Vertex Cover

What is the *smallest* Independent Set for this graph?



$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

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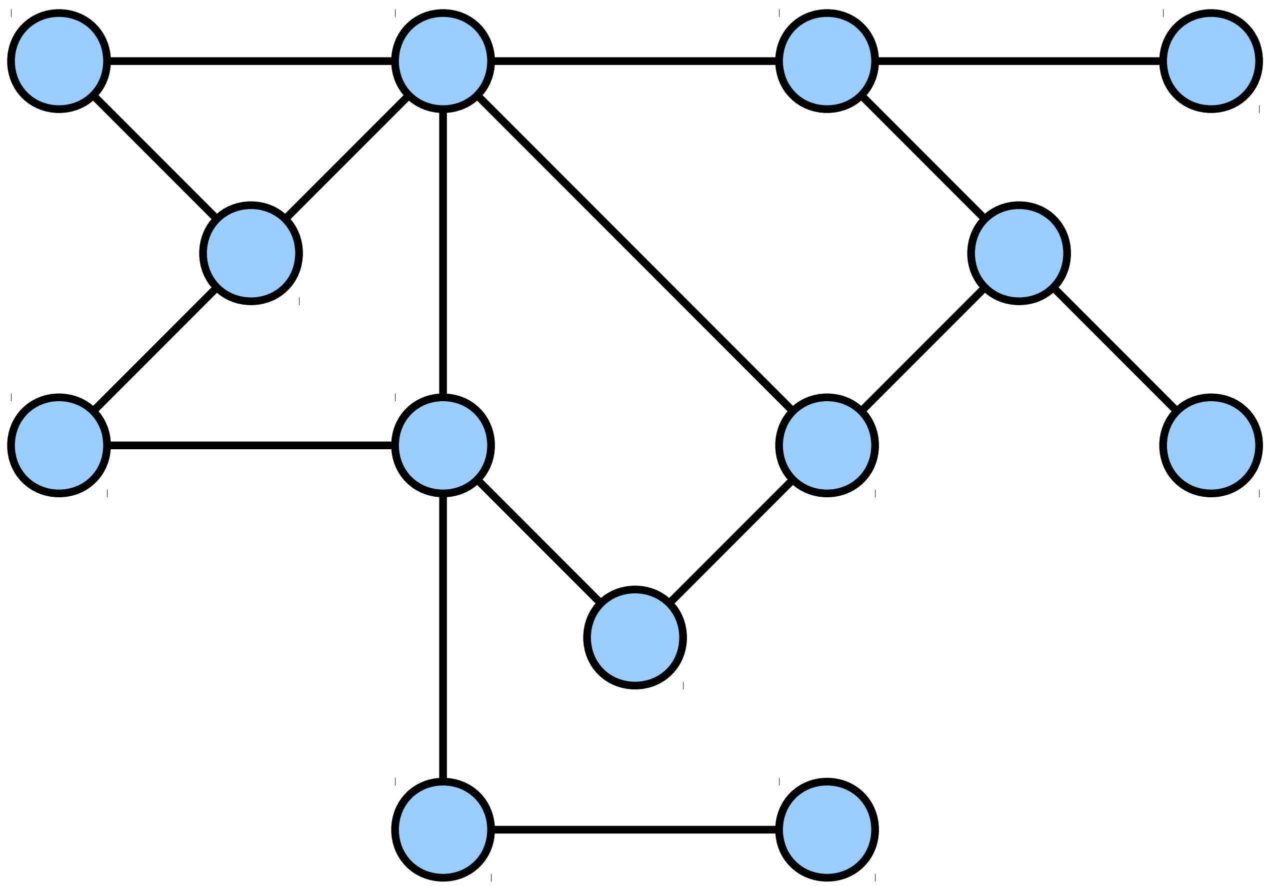
What is the *largest* Vertex Cover for this graph?



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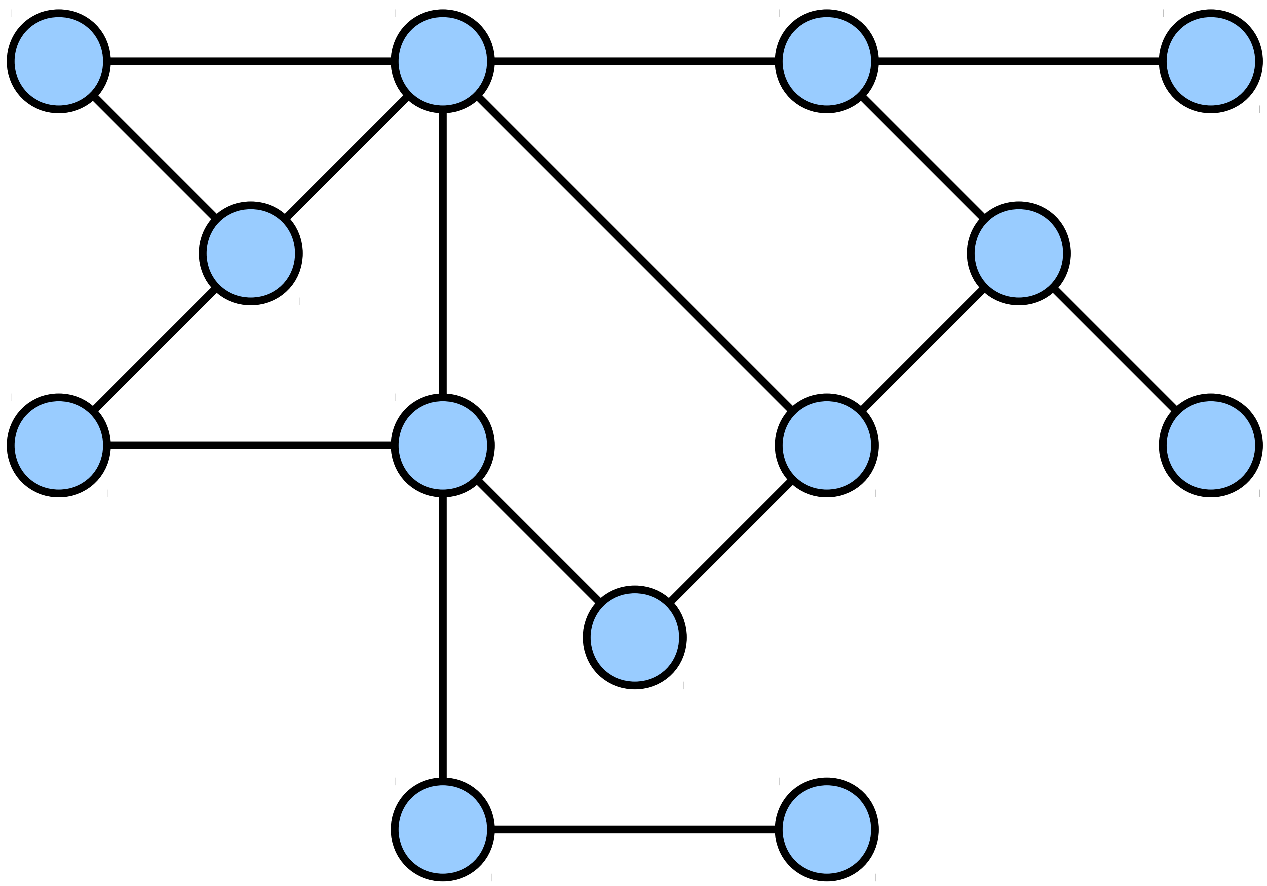
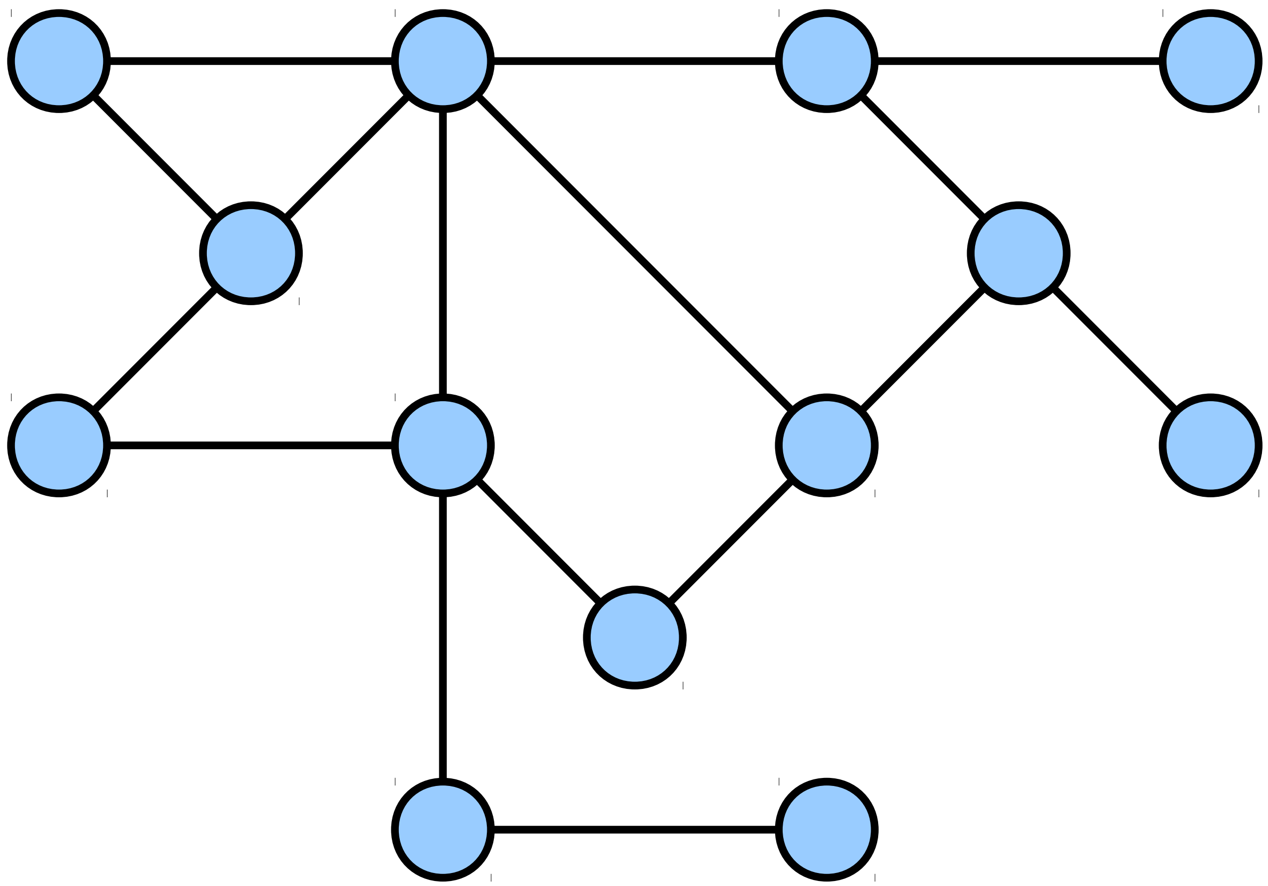
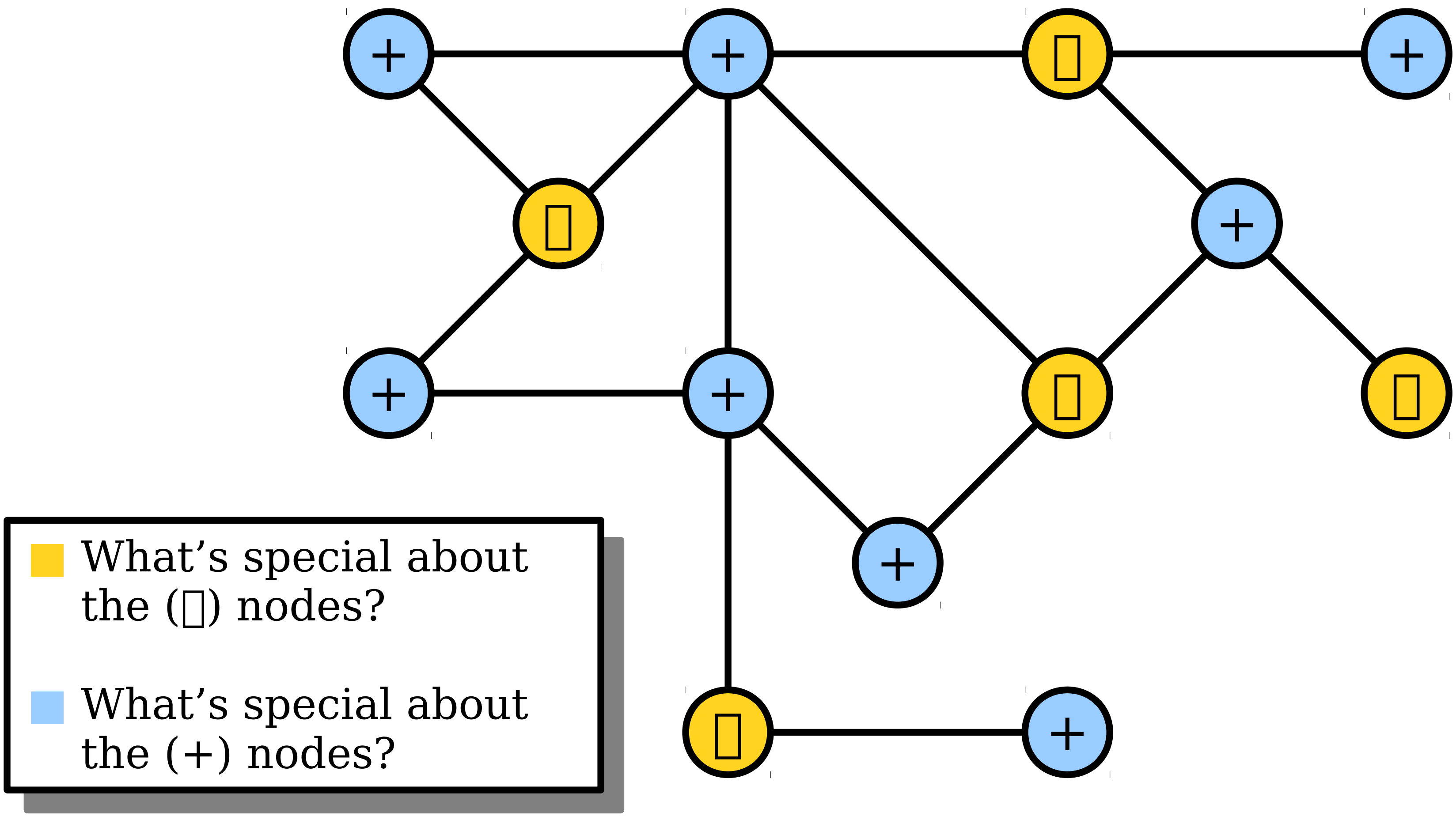


What is the *largest* Independent Set for this graph?
What is the *smallest* Vertex Cover for this graph?

A Connection



Independent sets and vertex covers are related.



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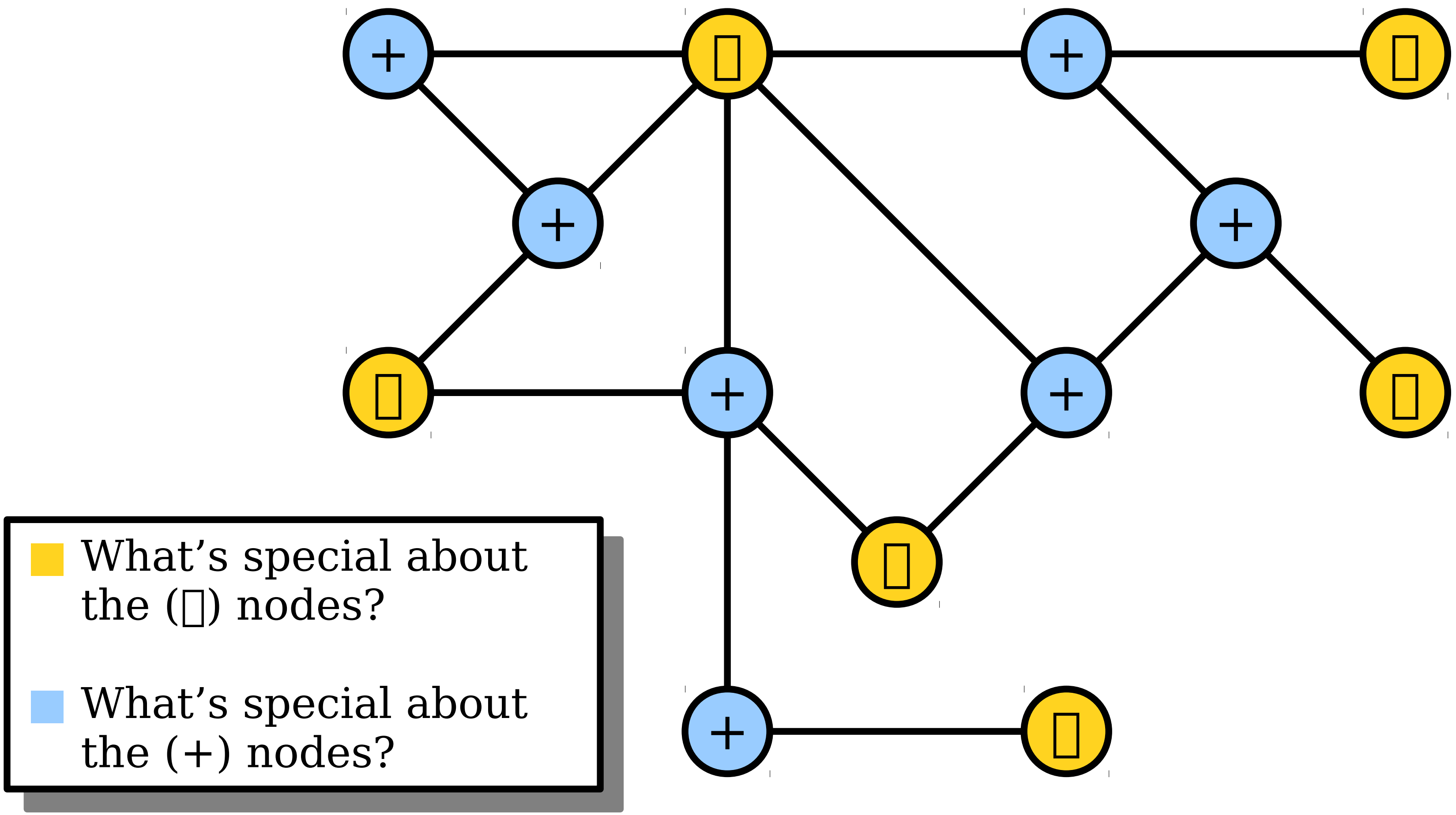

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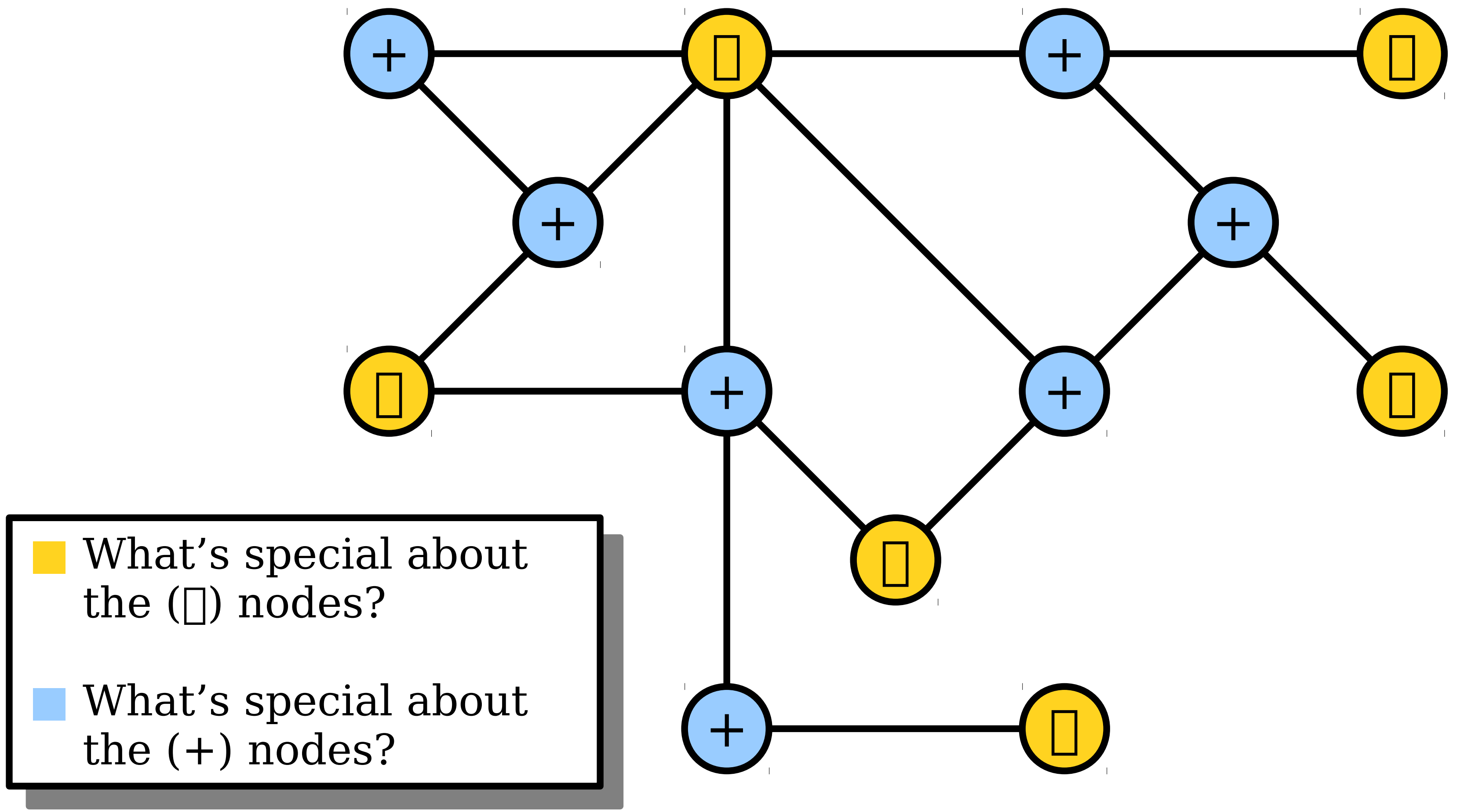

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G.

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How do we prove a **biconditional**?
Separately prove the forward and reverse directions of implication.

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G **if and only if** $V - C$ is an independent set in G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $V - C$ is an independent set of G, then C is a vertex cover of G.

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Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is **not** a vertex cover of G, then $V - C$ is **not** an independent set in G.

It turns out Lemma 2 is easier to prove in its contrapositive form.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$	$\forall y \in V - C.$
$u \in C \quad \lor \quad v \in C$	$\{x, y\} \notin E.$
$)$	

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What We're Assuming

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C is a vertex cover of G.

$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow$
$\quad u \in C \quad \lor \quad v \in C$
$)$

We're assuming a universally-quantified statement. That means we ***don't do anything right now*** and instead wait for an edge to present itself.

What We Need To Show

$V - C$ is an independent set in G.

$\forall x \in V - C.$
$\quad \forall y \in V - C.$
$\quad\quad \{x, y\} \notin E.$

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

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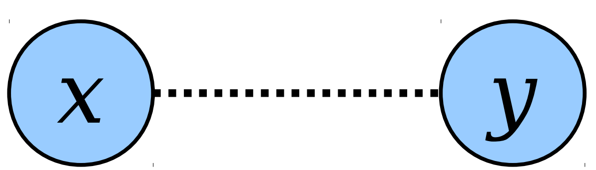
We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to$	$\quad \forall y \in V - C.$
$\quad u \in C \quad \lor \quad v \in C$	$\quad\quad \{x, y\} \notin E.$
$)$	
$x \in V - C.$	
$y \in V - C.$	



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Proof: Assume C is a vertex cover of G.

There's no need to introduce G or C here. That's done in the statement of the lemma itself.

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Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

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Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G **if and only if** $V - C$ is an independe

Lemma 1: done!
Now Lemma 2.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $V - C$ is an independent set of G, then C is a vertex cover of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is **not** a vertex cover of G, then $V - C$ is **not** an independent set in G.

To proceed, we need to take the negations of the FOL definitions of vertex cover and independent set.

Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

$$
\begin{aligned}
&\forall u \in C.\ \forall v \in C.\ (\{u, v\} \in E \to \\
&\quad u \in C \quad \lor \quad v \in C \\
&)
\end{aligned}
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- This says "there is an edge where both endpoints aren't in C."

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

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We're assuming an existentially-quantified statement, so we'll ***immediately*** introduce variables u and v.

We're proving an existentially-quantified statement, so we ***don't*** introduce variables x and y. We're on a scavenger hunt!

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Any ideas about what we should pick x and y to be?

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Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$.

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This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G, as required.

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Recap for Today

- A ***graph*** is a structure for representing items that may be linked together. ***Digraphs*** represent that same idea, but with a directionality on the links.
- Graphs can't have ***self-loops***; digraphs can.
- ***Vertex covers*** and ***independent sets*** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
  - Walking from one point to another.
- ***Indegrees and Outdegrees***
  - Counting how many neighbors you have, in the directed case.